A FRAMEWORK FOR SPATIAL POLITICAL ANALYSIS

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ABSTRACT. In order to understand current political events, it appears that the multidimensionality of political ideology is essential. Political candidates and parties differ not only in the traditional left-right dimension, but also concerning environmental issues, gender roles, law and order, migration, defense, etc. In addition, an arguably important phenomenon in democracies is abstention from voting because of alienation. We here outline a mathematical framework for the analysis of direct and representative democracy, political competition, political power, and coalition formation in a multidimensional setting with endogenous voter abstention. Our approach combines elements from non-cooperative game theory, cooperative game theory, geometry, and random-utility theory. We illustrate the framework by numerical examples and by applying it to data for the Swedish parliament.

1. INTRODUCTION

There are five main motivations behind this research project: (I) the importance of more than one dimension when representing ideologies of political parties, candidates, and voters, (II) the importance of endogenous voter abstention due to alienation, (III) the importance of more than two political parties or candidates, (IV) the relevance of vote shares—rather than numbers of votes—for political outcomes, and (V) the relevance of political parties' voting power in parliamentary decision-making.

To the best of our knowledge, there exists no mathematical modeling framework that addresses more than a few of these aspects. A pioneering work that incorporated aspects (I)-(III) is Davis, Hinich, and Ordeshook (1970). However, they did not

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analyze vote-share maximization or political power in parliaments. The only model of vote-share maximization we know of is that of Krasa and Polborn (2012). However, their analysis is limited to electoral competition between only two candidates, and in their model voters abstain only when indifferent between the two candidates. By contrast, in the present framework voters may abstain from voting if they feel ideologically distant from the political platforms proposed by candidates or parties.

An early comparative analysis of different motivations for political agents was given by Hinich and Ordeshook (1970). They compared the outcomes under vote maximization with outcomes under plurality maximization in electoral competition between two candidates. Wittman (1983) and Lindbeck and Weibull (1993) analyzed competition between two policy motivated candidates or political parties who also have an interest in winning *per se*. Laslier (2005) and Patty (2007) compared vote maximization with probability-of-winning maximization in a multiple-candidate setting without abstention. Hinich and Ordeshook (1969), Hinich, Ledyard, and Ordeshook (1972), Ledyard (1984), and Llavador (2000) allowed for voter abstention due to alienation, but they did not consider vote-share maximization.

We here outline a modeling framework that incorporates all five mentioned aspects under different forms of government: direct democracy, autocracy, presidential democracy, and representative democracy. The electorate is modeled as a heterogeneous continuum of voters. There are finitely many political agents, who may be political parties or individual political candidates. Every voter has a fixed political ideology, represented as a point in a multidimensional space. After the appointment of legislators—a president, governor, parliament or committee—the society faces randomly drawn political issues. On each issue the legislators need to choose a policy on a one-dimensional scale over which voters have single-peaked preferences. A voter's ideal policy on a given issue is determined by the voter's ideology. Political agents also have fixed "true" ideologies, also represented as points in the multidimensional ideology space. However, in electoral campaigns, they may commit to any platform they like. Also these platforms are represented as points in ideology space. Under representative democracy, legislators are chosen by the citizens. In presidential elections, exactly one candidate is elected, while in parliamentary elections multiple parties may obtain seats in parliament. Once elected, the president is assumed to single-handedly make decisions on all issues that arise during his or her mandate period and is then assumed to act consistently from his or her election platform.[1] Likewise, once the members of a parliament have been elected, they make policy decision by majority rule in parliament on all issues that arise, based on their platforms. By contrast,

[1] Generally, presidential systems are defined as a form of government where the legislative branch is separated from the executive branch led by the president. Our framework does not make the distinction between executive and legislative functions.

under direct democracy, all voters are invited to vote on each issue when it arises. In autocracy, finally, a single political agent has exogenously been given all power as legislator, to decide on all upcoming issues. The autocrat is assumed to make decisions according to his or her "true" ideology.

The analytical machinery in this framework combines elements from four separate literatures: (a) spatial models of electoral competition, pioneered by Hotelling (1929) and Downs (1957) and the above-mentioned multidimensional models,[2] (b) cooperative game theory, in particular an apparently not well-known spatial model developed by Lloyd Shapley in a 1977 RAND memorandum, as a generalization of the classical Shapley value,[3] (c) recent results in geometry concerning medians in multidimensional spaces (Durocher and Kirkpatrick, 2009, Durocher, Leblanc and Skala, 2017), and (d) a recent result in random utility theory (Fosgerau et al., 2018) that allows for outside options.

Concerning (a), spatial models of electoral competition, we recall that in Hotelling (1929) there are two competitors and a continuum of consumers or voters uniformly distributed over the unit interval. This model was followed up in political science by Downs (1957) and later extended to multidimensional settings by Davis and Hinich (1966) and followers. Hotelling assumed that each consumer buys exactly one unit or, equivalently, that each voter casts exactly one vote. Hence, no abstention. Smithies (1941) generalized Hotelling's model to allow for elastic demand or, expressed in terms of political competition, endogenous abstention from voting due to alienation. See Fournier, van der Straaten, and Weibull (2022) and the references therein for recent contributions to that literature.

Concerning (b), cooperative game theory, in Shapley's 1977 RAND memorandum suggested that finitely many players, such as political parties in a parliament under majority rule, may be identified with points in a multidimensional space of characteristics, representing their ideological positions. He suggested that political issues can be represented by real-valued linear functions on this space, with the interpretation that the value of such a function at any point in the ideology space represents the opinion that a player at that point has on that issue. The higher function value, the stronger policy measure the player would prefer on that issue. Hence, while ideology in Shapley's model is multidimensional, each issue induces a unidimensional ordering of the players' preferred policies on that issue. By assuming that all issues are equally likely, Shapley (1977) defined a player's power as the probability of being pivotal on

[2] The literature is vast, and includes Davis and Hinich (1966, 1967); Hinich and Ordeshook (1970); Davis, DeGroot, and Hinich (1972); McKelvey (1976, 1979); McKelvey and Schofield (1987); and Coughlin (1992).

[3] The spatial power index in Shapley (1977) was, in turn, inspired by a similar idea in Owen (1971) and was further developed by Owen and Shapley (1989). Again, the literature on cooperative game theory is vast.

a randomly drawn issue. The classical Shapley value is then obtained as the special case when players are symmetrically located in the space. We slightly extend that spatial power index and, by means of recent results in geometry, show that this index has appealing operational properties. In addition, we suggest an accompanying index of empowerment. This index quantifies the relative power gain or loss that a coalition bestows its members, compared with the power these players have if no coalition is formed. The empowerment index is very close in spirit to that of Alonso-Meijide and Carreras (2011). However, while their index is based on the classical Shapley value and defined in absolute terms, our index is based on the mentioned spatial Shapley value and is defined in relative terms.[4]

Concerning (c), recent work in geometry, Durocher and Kirkpatrick (2009) introduced a generalization of the univariate median to higher dimensions, called *the projection median*. The basic idea is to consider projections of a finite set of (weighted) points in a multidimensional space to straight lines, identify the median(s) on that straight line, and then take the average of these projections as lines, in all directions through the origin, are uniformly randomly drawn. Each point has a fixed positive weight when the projection median is calculated, just as political parties have fixed representations in parliaments.[5] The main result in Durocher, Leblanc and Skala (2017) is a characterization of this projection median, for finite weighted sets, as a certain average, calculated much along the same lines as Shapley's (1977) spatial power index . We use this characterization result when analyzing decision-making in parliaments under majority rule, and establish the connection with Shapley's (1977) spatial power index. We also extend the definition of the projection median from finite to (certain) infinite sets and relax the assumption of uniform random draws of straight lines (or "issues") to arbitrary probability distributions.

Concerning (d), we view voters' decisions, when voting in representative democracy on political candidates or parties, as discrete choices with two utility component. One deterministic geometric disutility component that represents disagreement with political agents' policies on issues, derived from their platforms as outlined above. The second component is a random variable that represents voters' idiosyncratic preferences concerning fixed traits of the political agents in question, such as their personality or record.[6] This is in line with Lindbeck and Weibull (1987, 1993), Coughlin (1992), and Banks and Duggan (2005). The random utility associated with abstention may reflect idiosyncratic costs or inconveniences associated with voting.[7]

[4] For more subtle and complex indices of empowerment, see Owen (1977), Kalai and Samet (1987), and Béal et al. (2018).

[5] Thus, the finite set of points is a so-called multi-set.

[6] The record can refer to anything that is considered fixed and given by voters, such as earlier achievements and failures, positions held, competence, honesty, etc.

[7] To be more precise, no voter faces random utility, but for any given ideology, voters that

The random utility vector, incorporating all choice alternatives, is said to have the invariance-of-the-maximum property (Fosgerau et al., 2018) if the distribution of the value of any specific alternative, conditional on that alternative being chosen, is the same, regardless of which alternative is considered. Endogenous voter abstention is conveniently included by invoking this invariance property.

As may already be apparent to the reader, the proposed analytical framework is too complex to permit analytical closed-form answers to many relevant questions. However, the framework is handy for numerical simulation. Hence, while we have analytical results only for some special cases, we illustrate the more general workings of the framework by way of numerical examples, and by applying it to data from the general parliamentary elections in Sweden in 2018 and 2022. We then use two data sets to represent the parties' locations in ideology space. The first data set is from Chapel Hill's expert survey (Jolly et al., 2019). Based on this data, we locate Swedish parties in a two-dimensional ideology space, a GAL-TAN diagram (Hooghe et al., 2002), where the horizontal axis is the traditional left-right ("economic redistribution") dimension, and the vertical axis represents positions on a composite scale from, broadly speaking, authoritarianism to liberalism. The second data set, SVT valkompass.svt.se, contains the responses that each political party gave before the election to a questionnaire of the Swedish National Broadcaster about their positions on 30 different issues. This second data set is used here as a robustness check for the two-dimensional representation, and also to illustrate the feasibility of numerical computation for high-dimensional ideology spaces within the framework.

The rest of the material is organized as follows. Section 2 presents the mathematical framework, sketches the political scenarios to be studied, and formalizes voters' and political agents' preferences. Sections 3-5 focus on the legislative stage. In Section 3, we apply the framework to direct democracy, while Section 4 analyzes decision-making under a single policy-maker, who may be an autocrat or an elected president. Decisions taken in parliament by majority rule are analyzed in Section 5, where we also define and analyze political power, consider coalition formation, and define the mentioned empowerment index for coalitions. Section 6 is devoted to political competition, both between candidates for a single office and between political parties for seats in a legislative body, such as a parliament. In Section 7 we apply the framework to the Swedish parliament, or "Riksdag". Section 8 concludes with a general discussion. More technical mathematical proofs are provided in an Appendix at the end.

share this ideology might have different idiosyncratic preferences or costs. The random term thus represents to the heterogeneity among voters with common ideology.

2. Framework

The framework presented in this paper permits the analysis of multidimensional public choice problems in societies under different choice procedures or *forms of government.* The society faces randomly drawn issues, and for each realized issue it decides on a policy. In the first stage, before any issue is drawn, a subset of citizens is designated as *legislators.* In the second stage, the *legislative stage*, issues appear. For each issue, a policy position is then chosen according to a choice rule for the legislature in question. This choice rule depends on the form of government, and it selects a unique policy for each issue that arises depending on the general political positions of the members or the legislature, to be called the *collective policy position* on that issue. A *form of government* thus in effect consists of two choice rules, one for each stage; by first assigning legislators and then, given the legislators, assigning a collective policy positions on each issue.

The society consists of a continuum of *citizens* and finitely many *political agents*, $i \in I = \{1, ..., n\}$. Such an agent may be a political candidate or a political party. In line with the literature following the seminal paper by Davis and Hinich (1966), the analysis will be carried out in an abstract *ideology space*, which we here take to be a d-dimensional Euclidean space, $\mathbb{R}^d$, for some positive integer d. Each citizen-voter and political agent has a fixed *ideology,* $x \in \mathbb{R}^d$.[8] The ideology distribution among the voters, the *voter distribution*, is represented by a probability density function $\phi : \mathbb{R}^d \to \mathbb{R}_+$ with convex support $X \subseteq \mathbb{R}^d$ on which it is continuous. Let $x^i \in X$ be the ideology of political agent i. For a set $I = \{1, ..., n\}$ of political agents, the associated vector $\vec{x} = (x^1, ..., x^n)$ will be called the (true) *ideology profile* of the set.

2.1. Geometry. Following Shapley (1977), *political issues* will be represented by straight lines through the origin $\mathbf{0} = (0, 0, ..., 0) \in \mathbb{R}^d$. The *policy position* of a voter or political agent with ideology $x \in X$ *on any issue* is defined by the orthogonal projection of x on the straight line $L \subset \mathbb{R}^d$ that represents the issue. This way, the ideology of a voter or political agent uniquely determines the policy position of this voter or agent on each issue.

More precisely, let $\Theta = \left\{\theta \in \mathbb{R}^d : \|\theta\| = 1\right\}$ be the *unit sphere* in $\mathbb{R}^d$, where $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{R}^d$. The elements θ of Θ are thus the *unit vectors* in $\mathbb{R}^d$. Let $\mathcal{L}$ be the set of all straight lines L through the origin. Any unit vector $\theta \in \Theta$ uniquely defines a straight line L_θ in $\mathcal{L}$, and hence a political issue:

$$L_\theta = \left\{x \in \mathbb{R}^d : x = t\theta \text{ for some } t \in \mathbb{R}\right\} \tag{1}$$

[8]By treating political agents in the same way as voters, the framework allows for endogenization of the set of political agents, in line with citizen-candidate models, as pioneered by Osborne and Slivinski (1996) and Besley and Coate (1997).

The *policy position* of an ideology $x \in X$ on an issue $L_\theta \in \mathcal{L}$ is given by the orthogonal projection of ideology x on L_θ:

$$p_\theta(x) = t_\theta(x)\,\theta \in \mathbb{R}^d \tag{2}$$

where $t_\theta(x)$ is the usual scalar (or inner) product between the vectors θ and x:

$$t_\theta(x) = \theta \cdot x = \textstyle\sum_{j=1}^{d} \theta_j x_j \in \mathbb{R} \tag{3}$$

From the vantage point of ideology $x \in X$ this is the *ideal policy* for issue L_θ. If an issue line passes through an ideology x, then the policy position of that ideology on that issue is x, and the absolute value of the ideal policy on the issue is $\|x\|$:

$$x \in L_\theta \qquad \Rightarrow \qquad p_\theta(x) = x \text{ and } |t_\theta(x)| = \|x\| \tag{4}$$

The magnitude of the *policy difference* between any two ideologies $x, x' \in X$ on any issue $\theta \in \Theta$ is defined as the distance between their policy positions on the straight line L_θ representing the issue, or, equivalently, as the difference between their ideal policies for the issue:

$$\|p_\theta(x) - p_\theta(x')\| = |t_\theta(x) - t_\theta(x')| \tag{5}$$

Remark 1. *For any unit vector $\theta \in \Theta$, its "mirror" $-\theta \in \Theta$ defines the same straight line; $L_{-\theta} = L_\theta$. However, this is inconsequential for the subsequent analysis, since the definitions of policy position and policy distance are independent of which of the two unit vectors is used.*[9] *Therefore we will use $L_\theta \in \mathcal{L}$ and $\theta \in \Theta$ interchangeably as representations of issues.*

Example 1. *Figure 1 depicts the ideologies of three citizens in two-dimensional ideology space. The horizontal axis may represent attitudes towards degrees of market regulation, with more regulation to the left and less regulation to the right (in line with conventional left-wing and right-wing economics attitudes). The vertical axis may represent attitudes towards degrees of environmental protection, with more protection in the upward direction and less protection in the downward direction. These two axes may thus be thought of as two issues, "market regulation" and "environmental protection", respectively.*

The left panel shows that while citizen A is the most left-wing in terms of market regulation, A is also the least environmentally concerned. By contrast, citizen B is the most environmentally concerned but takes an intermediate position on market

[9] To see this, note that $t_{-\theta}(x) = -t_\theta(x)$, and thus $p_\theta(x) = t_\theta(x)\,\theta = t_{-\theta}(x)(-\theta) = p_{-\theta}(x)$. Moreover, the policy distance is defined in absolute terms.

regulation, relatively close to that of A. Citizen C is the most right-wing concerning market regulation but has an intermediate opinion concerning environmental protection, relatively close to that of B. The right panel shows the three citizens' policy positions on a third issue, represented by the tilted straight line. This line may be thought of as representing the issue of carbon taxation, with higher taxes in the Northwest direction. Despite their similar concern for the environment, B and C take opposite policy positions on this third issue, with citizen A taking an intermediate position.

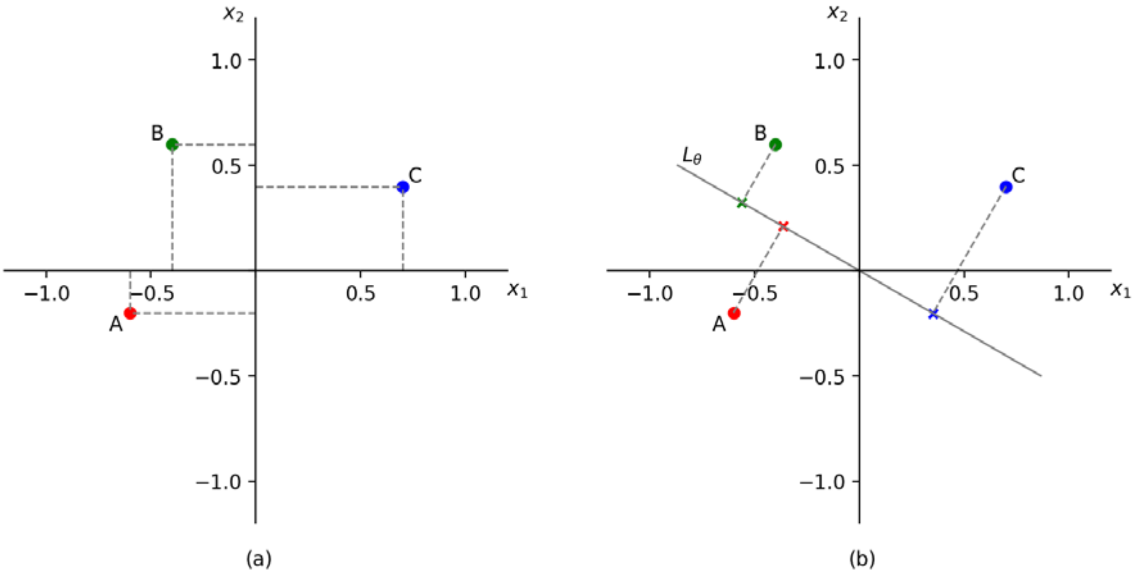

Figure 1: Three ideologies and their positions on three issues.

By the *policy trajectory* of any ideology we mean the set of policy positions of the ideology on all issues:

Definition 1. *The* policy trajectory *of any ideology $x \in \mathbb{R}^d$ is the subset $P(x) \subset \mathbb{R}^d$ defined by*

$$P(x) = \left\{x' \in \mathbb{R}^d : x' = p_\theta(x) \text{ for some } \theta \in \Theta\right\} \tag{6}$$

It may be noted that the origin of the ideology space belongs to the policy trajectory of every ideology, because for every ideology x there exists an orthogonal issue θ, and then $t_\theta(x) = \theta \cdot x = 0$. Hence $p_\theta(x) = \mathbf{0}$, so $\mathbf{0} \in P(x)$. Moreover, every ideology belongs to its policy trajectory. To see this, note that $\mathbf{0} \in P(\mathbf{0})$, and if $x \neq \mathbf{0}$ and $\theta = x/\|x\|$, then $t_\theta(x) = \|x\|$.

Basu et al. (2012) prove that $P(x)$ is always a sphere (circle when $d = 2$), with center at $\frac{1}{2}x$ and diameter $\|x\|$:[10]

Lemma 1 [Basu et al. 2012]**.** *For any* $x \in \mathbb{R}^d$:

$$P(x) = \left\{ x' \in \mathbb{R}^d : \left\| x' - \frac{1}{2}x \right\| = \frac{1}{2}\|x\| \right\} \tag{7}$$

Figure 2 shows the ideologies of three citizens (voters or political agents), the same as in Example 1, and their circular policy trajectories in two-dimensional ideology space. The straight line represents an issue, and the crosses on this line are the three citizens' policy positions on that issue.

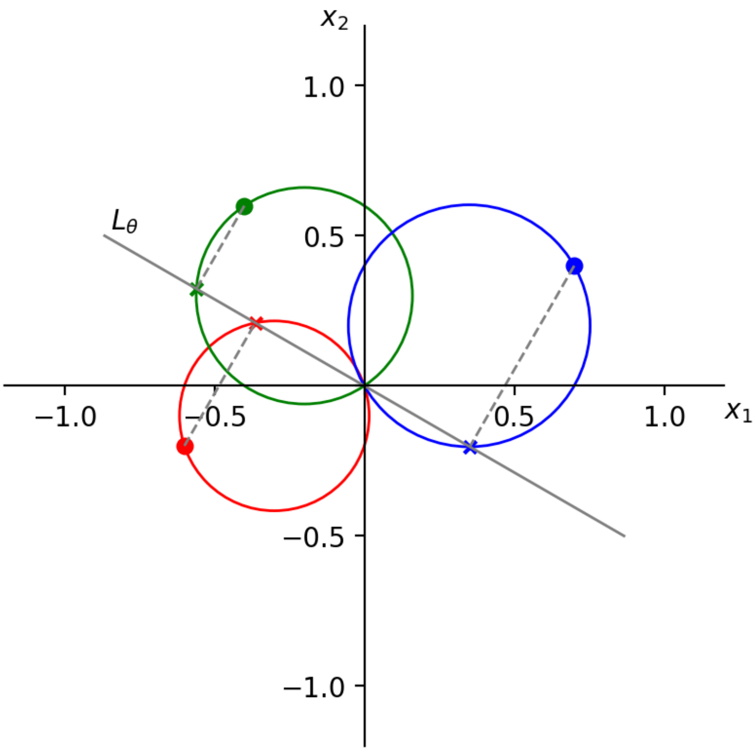

Figure 2. The policy trajectories of three ideologies.

Next, we provide a few definitions of median. First, a *median* of a univariate cumulative distribution function (CDF) $F : \mathbb{R} \to [0, 1]$ is any point $t^{med} \in \mathbb{R}$ such

[10]They use the term "projection sphere" to refer to this set.

that
$$\int_{\left(-\infty,t^{med}\right]} dF\left(t\right) \geq \frac{1}{2} \quad \text{and} \quad \int_{\left[t^{med},+\infty\right)} dF\left(t\right) \geq \frac{1}{2} \tag{8}$$

There always exists at least one median, and if there are multiple, then they constitute a closed and bounded interval, $T^{med} = \left[t_{-}^{med}, t_{+}^{med}\right]$, the interior of which has measure zero with respect to the corresponding probability distribution. Evidently, a probability distribution F that is continuous and positive has a unique median. In case of multiple medians, we will henceforth for definiteness (and in line with Durocher and Kirkpatrick, 2009, see below) choose the midpoint of the interval T^{med} as *the* median, and denote it τ^{med}.

In higher dimensions, one defines the median in any given direction by applying the above definition to the marginal distribution in that direction. Let $F : \mathbb{R}^d \to [0,1]$ be a CDF, and let $\theta \in \Theta$. The *marginal probability distribution* $F_\theta : \mathbb{R} \to [0,1]$ *in direction* θ is defined by

$$F_\theta\left(t\right) = \int_{\theta \cdot x \leq t} dF(x) \quad \forall t \in \mathbb{R} \tag{9}$$

Definition 2. *A* ***median*** *of a CDF* $F : \mathbb{R}^d \to [0,1]$ ***in direction*** $\theta \in \Theta$ *is any point* $x_\theta^{med} = t^{med}\theta \in L_\theta$ *such that* $t^{med} \in \mathbb{R}$ *is a median of* F_θ*.*

As above, the set of median points in any given direction $\theta \in \Theta$ constitutes a closed and bounded line segment, $\left[x_{\theta-}^{med}, x_{\theta+}^{med}\right] = \left[t_{-}^{med}\theta, t_{+}^{med}\theta\right]$, and we refer the midpoint of this line segment as *the* median in direction θ, and denote it $x_\theta^{med} \in L_\theta$ We note that the median for $-\theta$ is the same as for θ.

As a generalization of the univariate median to higher dimensions, Durocher and Kirkpatrick (2009) define the *projection median* of a CDF F with finite support in a Euclidean space with dimension $d > 1$ as a normalized average of the median positions when issues are uniformly randomly drawn from the unit sphere Θ. Let κ be the uniform probability distribution on Θ (formally, the Haar measure):[11]

Definition 3 [Durocher and Kirkpatrick, 2009]**.** *The* ***projection median*** *of a CDF* $F : \mathbb{R}^d \to [0,1]$ *with finite support is the point*

$$x^{med} = d \int_{\theta \in \Theta} x_\theta^{med} d\kappa\left(\theta\right) \tag{10}$$

where $x_\theta^{med} \in L_\theta$ *is the median of* F *in direction* θ*.*

[11] The mapping $\theta \mapsto x_\theta^{med}$ has finitely many discontinuities and is thus integrable w.r.t. κ.

The normalization factor d, the dimension of the space, may appear surprising. It is related to the fact that for each direction $\theta \in \Theta$ there are $d-1$ orthonormal unit vector pairs that skew the average towards the origin. To see this in two-dimensional space, note that the projection median of a CDF F that places unit mass on a single point $x \in X$ is the point x itself, while without the normalization it would have been $\frac{1}{2}x$.[12]

Unlike other generalizations of the univariate median, the projection median meets the desiderata of being *invariant under translation, uniform rescaling, and change of basis* of the space.[13] It is also *subspace consistent* in the sense that if the finite support belongs to a subspace or hyperplane of the original Euclidean space, then also the set's projection median belongs to that subspace or hyperplane and, moreover, coincides with the projection median as defined for the probability as defined over that subspace or hyperplane. Moreover, the projection median belongs to the convex hull of the finite set of points, and it has appealing robustness and computational properties (see Basu et al., 2012, and Durocher, Leblanc and Skala, 2017).

Remark 2. *The subsequent analysis is invariant under such isometric transformations as reflection, rotation, and translation. Moreover, apart from policy distances, the analysis is also invariant under such angle-preserving transformations as rescaling of the axes with unequal scale factors. Under transformations that preserve angles but not distances, policy distances do change, but ordinal rankings of policies on any given issue are invariant, as are the probability measures on the unit sphere.*

2.2. Forms of government. As mentioned in the introduction, we will consider four forms of government: direct democracy (D), autocracy (A), presidential democracy (P), and parliamentary democracy (Q). Each form defines the rules in each stage of the two-stage choice procedure.

The first-stage rules under direct democracy and autocracy are trivial. In both cases, the legislature is determined exogenously; either all citizen or exactly one citizen. Hence, under direct democracy the entire electorate together constitute the legislature, while under autocracy a single citizen, the autocrat, is exogenously appointed legislator. Under presidential and parliamentary democracy, however, the legislators are chosen by voters. Each political agent seeking office—a candidate for presidency or a political party for election to parliament—makes a binding commitment to an ideology in ideology space, a *political platform*, $y^i \in X$, which may, but

[12] It may also be noted that for $d = 1$, the projection median is the usual uni-dimensional median. For then there are only two unit vectors, $\theta = 1$ and $\theta = -1$, and $x_1^{med} = x_{-1}^{med} = \tau^{med}$.

[13] More exactly, by "translation" and "uniform rescaling" we mean transformations of the form $x \mapsto \alpha x + y$ for some $\alpha \neq 0$ and $y \in X$. By "invariant" we mean that the projection median is likewise transformed ("equivariant"): if it was x^{med} before the transformation, it will be $\alpha x^{med} + y$ after the transformation.

need not, differ from the agent's true ideology $x^i \in X$. Let the associated platform profile be denoted $\vec{y} = (y^1, ..., y^n)$. Voters observe these platforms and either vote for one political agent or abstain. The outcome of the elections determines who wins the presidential election or what representation the political parties will have in parliament. In the first case, the candidate with the most votes wins.[14] In the second case we assume proportional representation and we abstract away from the discretization that occurs when vote shares are transformed to integer mandates. Hence, a party's share of seats in parliament is taken to be identical with its vote share in the election.

For the legislative stage under autocracy and presidential democracy, we make the heroic assumption that dictatorial rule is applied, that is, the policy position of the exogenously imposed autocrat or elected president determines the policy on every issue. The autocrat's decisions are given by his or her true ideology, while the president acts in accordance with the political platform announced in the electoral campaign. Under direct democracy and parliamentary democracy, policy decisions are taken issue by issue according to majority rule, applied directly to the whole electorate in the first case and to the elected parliament in the second case. By assumption, all voters and political agents (also those who have espoused a political platform y^i distinct from their true ideology x^i) have well-defined single-peaked preferences over the unidimensional space L_θ of policy alternatives concerning any issue $\theta \in \Theta$ at hand. As is well-known since Black (1948), the existence of a Condorcet winner is guaranteed under these circumstances. Therefore, without explicitly modeling the details of voting procedures, the collective policy position on each issue will be a Condorcet winner. How ties are broken will be defined subsequently. We use the term *majority rule* according to this definition.

For the four forms of government to be studied, the following terminology and notation will be used. The *collective policy trajectory* of a society under a given form of government $G \in \{D, A, P, Q\}$ is defined by

$$\tilde{X}^G = \left\{x \in \mathbb{R}^d : x = \tilde{x}_\theta^G \text{ for some } \theta \in \Theta\right\} \tag{11}$$

where $\tilde{x}_\theta^G \in \mathbb{R}^d$ is a *collective policy position* on issue θ, taken according to the choice procedure given by the form of government G in question. In most cases to be studied, the collective policy position on an issue will be unique.

2.3. Preferences over policies and policy positions.

The ideology $x \in X$ of a citizen — voter or political agent — defines this citizen's policy position $p_\theta(x) \in L_\theta$ and ideal policy $t_\theta(x) \in \mathbb{R}$ for each issue $\theta \in \Theta$, see

[14] In the case of a tie, the winner is chosen from among the candidates with the same maximal number of votes by way of a uniform random draw.

equations (2) and (3). We assume that the utility associated with implementation of any policy $t \in \mathbb{R}$ on the issue decreases with the squared distance from the citizen's ideal policy for that issue:

$$u_\theta(t, x) = -\lambda(\theta) |t - t_\theta(x)|^2 \quad (12)$$

where the function $\lambda : \Theta \to \mathbb{R}_+$ is strictly positive, bounded and continuous. The value $\lambda(\theta) > 0$ is thus the "weight" that citizens attach to policy differences on issue θ.[15] For every issue $\theta \in \Theta$, equation (12) defines a utility (or Bernoulli) function $u_\theta : \mathbb{R} \times X \to \mathbb{R}$. One obtains citizens' utility associated with any issue $\theta \in \Theta$ and any political agent's political platform as the weighted squared distance between the policy associated with the platform and the voter's own ideal policy:

$$U_\theta(y, x) = -\lambda(\theta) |t_\theta(y) - t_\theta(x)|^2 \quad (13)$$

For each issue $\theta \in \Theta$, this equation defines a utility function $U_\theta : X^2 \to \mathbb{R}$ for citizens with ideology x under the political agent's policy.

Remark 3. *For any fixed and given issue, (12) and (13) are equivalent with arbitrary utility functions that monotonically decrease with policy distance from the citizen's ideal policy. However, the mathematical form (including the weight function λ) of issue-based utilities matter in* ex ante *evaluations, that is, before issues are randomly drawn.*

Let μ be any probability measure on the unit sphere Θ. Applying expected-utility theory, the preferences of a citizen with ideology $x \in X$ over platforms $y \in X$, under any issue distribution μ, can be represented by the expectation[16]

$$\bar{U}_\mu(y, x) = \int_\Theta U_\theta(y, x)\, d\mu(\theta) \quad \forall x, y \in X \quad (14)$$

The next lemma establishes conditions under which this utility representation results in standard Euclidean preferences over ideology space. More precisely, if issues are uniformly random and all issues have the same utility weight, then the expected utility for a citizen with ideology x from the implementation of the policies of any ideology y is proportional to the squared Euclidean distance between the two ideologies. Let κ be the uniform distribution on Θ.

[15] Generalization to different weight functions for different citizens is feasible but notationally and analytically heavy.

[16] This expectation is finite since the policy trajectory of any ideology position is a bounded set (in fact, a sphere).

Lemma 2. *Suppose that $\lambda(\theta) = \bar{\lambda} > 0$ for all $\theta \in \Theta$. Then*

$$\bar{U}_\kappa(y,x) = -\frac{\bar{\lambda}}{2}\|x-y\|^2 \quad \forall x,y \in X \tag{15}$$

(Proof in the Appendix.)

For general issue distributions μ and general issue-weight functions λ, this equality does not hold, so in general, citizens with preferences (13) may well have non-Euclidean preferences over political agents' platforms.

Example 2. *Let $d = 2$, with a voter located at the origin, $x = (0,0)$, a policy-maker at $y = (0,2)$, indicated by a (blue) point, when the issue at hand is $\theta = \left(2/\sqrt{5}, 1/\sqrt{5}\right) \in \Theta$, indicated by the thin straight line in the diagram below. The (red) circle is the policy trajectory of the policy-maker. The policy distance between the voter and the policy-maker on issue θ is $|t_\theta(y) - t_\theta(x)| \approx 0.89$. This is the length of the straight thick (blue) line segment in the diagram. According to Lemma 2, the average squared length of this straight line, when the issue is uniformly randomly drawn, is 2.*

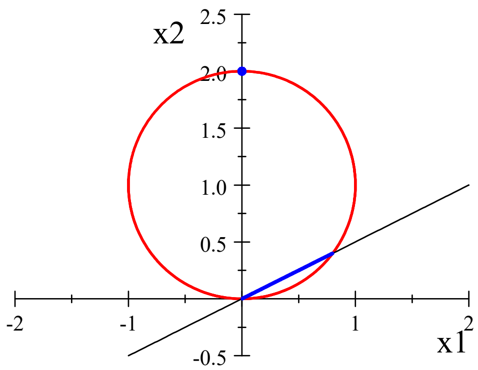

Figure 3: The policy trajectory of a decision-maker, and the distance between the policy positions of the DM and a voter's ideal policy on an issue.

In a continuum electorate, individual votes do not matter. In particular, strategic voting is not an issue.[17] One interpretation of the present approach is that voters feel a duty to vote on their most preferred alternative and weight this deontological motive against a personal cost or inconvenience of voting (c.f. Alger and Laslier, 2022).

[17] For an analysis of strategic voting in a finite electorate, see e.g. Krishna and Morgan (2012, 2015) and the literature cited therein.

2.4. Additional motives for voters. We consider two more motives for voters, one associated with non-ideological preferences for or against political agents, and one with cost of voting.

Under representative democracy, when voters delegate the decision-making to political agents, voters may have preferences over fixed traits of these political agents, such as their personalities or political records. In order to incorporate these, we follow the literature on probabilistic voting (see Lindbeck and Weibull, 1987, 1993, Coughlin, 1992, and Banks and Duggan, 2005), and assume that voters have additively separable preferences over combinations of policies and political agents. More precisely, the utility that a voter with ideology $x \in X$ associated with implementation of any policy $t \in \mathbb{R}$ by political agent $i \in I$ on any issue $\theta \in \Theta$ is

$$\tilde{u}_\theta(t, i, x) = u_\theta(t, x) + \zeta_i \tag{16}$$

where $u_\theta(t, x)$ is defined in (12), and ζ_i is the *valence* of political agent i. The valence may be deterministic or random, where the randomness does not necessarily mean that individual voters have randomly fluctuating preferences over political agents. Instead, we take it to represent voter heterogeneity, where each voter attaches a deterministic subutility to each political agent (a realization of the random variable ζ_i), but voters differ in this respect. While, in general, from the vantage point of a given ideology x, the probability distribution for a political agent's valence ζ_i may depend on x, we assume that the valence distribution for a political agent is the same all voters, and denote its CDF F_i.

It follows that a voter with ideology x attaches utility

$$\tilde{U}_\theta(y, i, x) = U_\theta(y, x) + \zeta_i \tag{17}$$

to any political platform $y \in X$ held by political agent $i \in I$ on any issue $\theta \in \Theta$, where $U_\theta(y, x)$ is defined in (13).

The cost of voting becomes relevant when voters consider whether to abstain instead of voting for one of the n political agents. The utility associated with abstention, the *abstention utility*, captures any such costs. It is treated as a random variable from the viewpoint of a political agent or outside observer. This random variable is denoted ε and has CDF $F_0 : \mathbb{R} \to [0, 1]$, which we assume is the same for all voters, irrespective of their ideology. The random variables, ζ_1,..., ζ_n, ε are assumed to be statistically independent, not only across the $n + 1$ choice alternatives faced by each voter, but also across the continuum of voters.

2.5. Additional motives for political agents. Just like voters, political agents have additional preferences, alongside their own "true" ideologies. The motives of real-world political agents, whether political candidates or parties, are varied and complex. An *opportunistic* political agent i chooses its platform y^i to maximize either its number of votes, its vote share, its expected power, or some mixture of these self-serving motives.[18] An *idealistic* candidate or party commits to its true ideology, x^i, even if unpopular. A *policy motivated* political agent i chooses its platform y^i to make the resulting implemented policy positions on issues as close as possible to its own "true" policy positions. In a parliamentary election, such a political party commits to a platform that it believes, on average, will move the policy decisions in parliament closest to its policy positions on issues that will appear (taking into account both the dependency of the election results on its platform and on its influence on decisions in parliament once elected). In the same vein, a policy-motivated presidential candidate will commit to a platform such that the policies taken by the winner of the election will be closest to the candidate's own policy positions.

Real-life political candidates and parties may have motives that are mixtures of all the above and may also contain other elements. However, here we will focus on a few pure cases.

3. Direct democracy

Under direct democracy, each citizen is a legislator and is invited to vote on issues as they arise. We here first assume full participation, and afterwards introduce endogenous abstention. On any given issue $\theta \in \Theta$ the collective policy position $\tilde{x}_\theta^D \in \mathbb{R}^d$ is taken to be a Condorcet winner. As stated in the previous section, there exists a Condorcet winner for each issue, since the policy positions on any issue are totally ordered and voter preferences are single-peaked. Moreover, due to the continuous and positive voter distribution, under full participation the Condorcet winner is unique and given by the median position in direction θ, as defined in Definition 2. Thus, the resulting collective policy position is such that half the electorate prefer "more" — a higher policy on the issue, and the other half "less" — a lower policy.

Likewise, the collective policy trajectory under direct democracy, $\tilde{X}^D$, is given according to (11). These are the policy positions taken by the society on all issues in the idealized setting where all eligible citizens are costlessly and perfectly informed about every proposed policy, a referendum is costless to organize, participation in a referendum is costless to each citizen, and all citizens vote on all issues. The set $\tilde{X}^D$ may thus serve as a benchmark for what collective decisions in an "ideal" democracy will lead to. This set is mathematically well-behaved.

Lemma 3. *For any continuous PDF ϕ with convex support X, the function $\theta \mapsto \tilde{x}_\theta^D$ is continuous, and the policy trajectory $\tilde{X}^D \subseteq X$ is a compact and connected set.*

[18]In the case of only two political parties, each party may also strive to maximize its plurality, the difference in votes, see Section 6.1.

(Proof in Appendix.)

For some distributions, the collective policy trajectory of the electorate coincides with the policy trajectory of the ideology of a single citizen. This is the case when the voter distribution ϕ is centrally symmetric, by which is meant that there exists an ideology $x^o \in X$ such that $\phi(x^o - x) = \phi(x^o + x)$ for all $x \in \mathbb{R}^d$. Clearly the policy position of ideology $x^o \in X$ will then be the median on any issue $\theta \in \Theta$. Formally:

Proposition 1. *For any PDF ϕ that is centrally symmetric around some ideology $x^o \in X$, its collective policy trajectory is the policy trajectory of ideology x^o, $\tilde{X}^D = P(x^o)$.*

The next example illustrates how ϕ may fail to be centrally symmetric in a potentially realistic way. The political science literature often discuss the possibility of a so-called "inverted U" or "horseshoe" distribution, for example in the context of Euroskepticism and the attitudes towards redistribution (Hooghe et al., 2002).

Example 3. *Consider the case of $d = 2$, as in a GAL-TAN diagram, and a voter distribution ϕ of the form*

$$\phi(x) = \frac{1}{4\pi\sigma_1\sigma_2} \exp\left(-\left(\frac{x_1 - \mu_1}{2\sigma_1}\right)^2\right) \exp\left(-\left(\frac{x_2 - a + b(x_1 - \mu_1)^2}{2\sigma_2}\right)^2\right) \tag{18}$$

This is the PDF of a random vector $Z = (Z_1, Z_2)$ where $Z_1 \sim N(\mu_1, \sigma_1^2)$ and, conditional on $Z_1 = x_1$, $Z_2 \sim N(a - bx_1^2, \sigma_2^2)$. In a GAL-TAN diagram, the marginal distribution along the horizontal left-right dimension is normal, and along the authoritarian-liberal dimension the marginal distribution is conditionally normal, given x_1, with a lower mean value as x_1 moves away from its mean value. A horse-shoe distribution where centrists in the left-right dimension are more liberal than extremists in the left-right dimension. The "muffin-shaped" policy trajectory $\tilde{X}^D$ is shown in the diagram below, along with a contour map of ϕ (thin), for $\mu_1 = 0$, $\sigma_1 = a = b = 1$, and $\sigma_2 = 1/2$. For comparison, the policy trajectory of a centrally symmetric distribution ($b = 0$) is also shown. Then ϕ is centrally symmetric around its mode $x^o = (\mu_1, a) = (0, 1)$, so the collective policy trajectory coincides with the policy trajectory of x^o.

Returning to the "horse-shoe" case $b = 1$, the median-voter position $\tilde{x}_\theta$ varies with the issue $\theta \in \Theta$. For θ horizontal, it is the projection of the mode of the distribution, $\tilde{x}_\theta = (0, 0)$, since in that direction the marginal distribution is symmetric. In the vertical direction, the marginal distribution is not symmetric, and its median position is lower than x^o (the projection of the mode on the vertical issue is itself). This explains why the policy trajectory is flatter than a circle. Moreover, ϕ is (mirror) *symmetric with respect to the vertical axis, so any two issues θ and θ' that make the same angle, only in absolute terms, with the vertical axis, have median positions $\tilde{x}_\theta = (x_1, x_2)$ and $\tilde{x}_{\theta'} = (x_1', x_2')$ with $x_1' = -x_1$ and $x_2' = x_2$, which explains the symmetry of the policy trajectory $\tilde{X}^D$ with the vertical axis. We finally note that the projection median of ϕ, shown as a black square, is approximately $x^{med} = (0, 0.47)$, a point on the vertical axis, below the mode x^o but above the policy trajectory.*

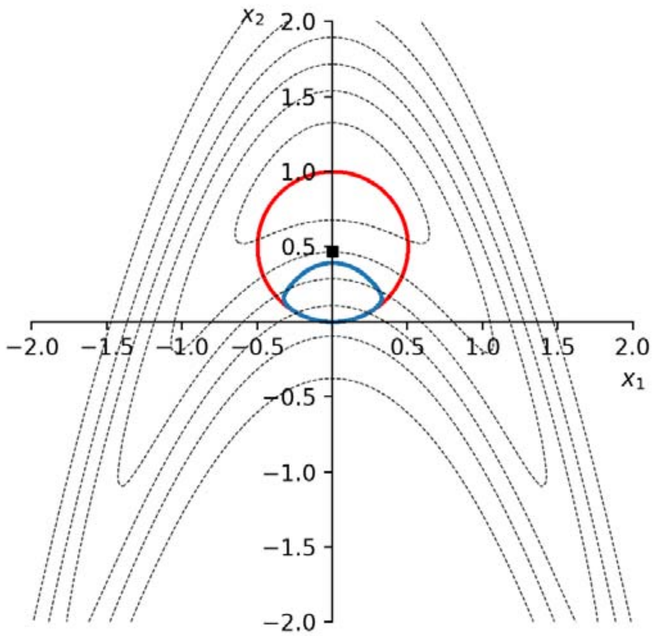

Figure 4. Projection trajectory of a horse-shoe distributed electorate.

3.1. Abstention.

In practice, not every voter participates in each referendum. Will the median voter position on each issue then still be a Condorcet winner? We here sketch an extension that allows for the possibility of abstention in a referendum. In order to analyze the set of Condorcet winners on an issue, we consider referenda in the form of a choice between two policy positions, $t_1\theta, t_2\theta \in L_\theta$ on some given issue $\theta \in \Theta$. For a voter with ideology $x \in X$, the utilities associated with the three alternatives are

$$V_\theta(t_1, t_2, x) = \begin{cases} u_\theta(t_1, x) & \text{when voting for } t_1\theta \\ u_\theta(t_2, x) & \text{when voting for } t_2\theta \\ \varepsilon & \text{when abstaining} \end{cases} \tag{19}$$

where the random variable ε is the abstention utility defined in Section (2.4). Each voter chooses the alternative with the highest (random) utility. Hence, a voter with ideology x abstains if

$$\varepsilon > -\lambda(\theta) \min\left\{ |t_1 - t_\theta(x)|^2, |t_2 - t_\theta(x)|^2 \right\} \tag{20}$$

Otherwise, the voter votes on the position closest to the voter's position on the issue. If the positions are equidistant from the voter's, then the voter casts a random vote on any one of them.[19] We assume that the random abstention utilities ε are statistically independent across voters. If the probability distribution of the abstention utility has full support on the real line, then the abstention probability is positive.

Accounting for endogenous participation in a referendum in this way, a policy position on an issue $\theta \in \Theta$ is a *Condorcet winner* if it obtains at least half the votes given against any contestant alternative on that issue. The following result provides sufficient conditions for existence and uniqueness of a Condorcet winner.[20]

Proposition 2. *If ϕ is unimodal and centrally symmetric around the location $x^o \in X$, and F_0 has full support, then the policy position $p_\theta(x^o) \in L_\theta$ of ideology x^o on any issue $\theta \in \Theta$ is the unique Condorcet winner on that issue.*

(Proof in Appendix.)

4. Autocracy and presidential democracy

Legislature under autocracy and presidential democracy is the same, once the autocrat or elected president has taken power. In both cases, the collective policy decisions are taken by one decision–maker. We will refer to this political agent as the policy-maker, whom we take to act according to some platform $y \in X$. In the case of autocracy, $y \in X$ is the autocrat's "true" ideology, while in the case of presidential democracy it is the platform to which the president has committed in the electoral stage. In both cases, the collective policy position for society is given by $p_\theta(y) \in L_\theta$ for any issue $\theta \in \Theta$ that arises, and the collective policy trajectory $\tilde{X}^A$ and $\tilde{X}^P$, respectively, is thus $P(y)$.

How do the policy positions decided by a single policy-maker compare with those decided under direct democracy for a society with a fixed voter distribution? An indicator of the welfare *loss* in such a society due to delegating policy decisions to a single policy-maker with platform $y \in X$, instead of applying direct democracy

[19] Since the voter distribution in ideology space is absolutely continuous, this random choice has no effect on voting outcomes.

[20] To be more precise, by *unimodality* is here meant that there exists a location $x^o \in X$ such that along any straight line through that point, the probability density $\phi(x)$ is increasing as x moves towards x^o from any side of x^o.

on each issue, is the average expected squared policy distance between the collective policy positions of two government types, where the average is taken over the whole electorate, with equal weight to all citizens:[21]

$$K(y) = \int_{\theta\in\Theta} \lambda(\theta) \left| t_\theta(y) - t_\theta\left(\tilde{x}_\theta^D\right) \right|^2 d\mu(\theta) \tag{21}$$

where $\tilde{x}_\theta^D \in L_\theta$ is the collective policy position on issue θ under direct democracy and μ is the probability distribution of issues.

Let $K^* = \inf_{y\in X} K(y)$, the greatest lower bound on the cost of delegation to a single citizen, to be called the *cost of delegation*.[22] It is clearly non-negative, and it is zero if the voter distribution ϕ is such that its policy trajectory $\tilde{X}^D$ is a sphere. More precisely, by Propositions 1 and 2:

Corollary 1. *$K^* \geq 0$ and $K^* = 0$ if ϕ is unimodal and centrally symmetric.*

The so calculated cost of delegation neglects voters' potential utility from the personality of the policy-maker. If ζ_i is a random variable representing the valence that voters attach to political agent $i \in I$, and this decision-maker is the autocrat or elected president, then

$$\int_{x\in X} \tilde{U}_\theta(y,i,x)\,\phi(x)\,dx = \mathbb{E}[\zeta_i] - K(y)$$

is a welfare measure that also includes voters' average utility from fixed traits of the decision-maker.

5. Parliamentary democracy

Under parliamentary democracy, voters elect members of a legislative body, a parliament, with a fixed and given number of seats. And thereafter political decisions are taken, on behalf of the voters, by the members of parliament. In the next section we look at the former, but for now we take the outcome of the election as given and focus on decision-making under majority rule in a parliament with given parties and mandates. For ease of exposure, we assume full participation, and that all legislators vote according to their party line.

Similar to direct democracy, but now the legislative body limited to the parties, on any given issue we take the collective policy position to be the Condorcet winner

[21] This disregards the large costs associated with organizing a referendum. Evidently, that cost should be taken into account in a realistic comparison between autocracy and direct democracy.

[22] Such a citizen can be seen as a benevolent dictator, one whose decisions are the most beneficial a society can get under one-person rule.

in parliament, when it is unique. Existence follows from the assumed single-peaked preferences of political agents over the one-dimensional space of policy alternatives for each issue. Since full participation is assumed, a Condorcet winner is a median voter on any given issue. Moreover since there are finitely many legislators with there can be at most two medians for each issue. In that case, we take the parliamentary policy position to be the midpoint of these two positions.[23]

To make this more precise, let the set of represented political parties be I. We assume that each party $i \in I$ during the election campaign has committed itself to an ideology, a platform $y^i \in X$ that may or may not coincide with the party's true ideology x^i. Let $\vec{y}$ denote the platform profile of the set I. Suppose that, associated with each party $i \in I$ there is a positive weight or *mass* $m_i > 0$, and let M be the total mass.[24] These weights may be the number of seats they each hold in parliament. Let $\Phi^I : \mathbb{R}^d \to [0,1]$ be the cumulative mass distribution function associated with platform profile $\vec{y}$. The parliament is thus defined by the triple $\langle I, \vec{y}, \Phi^I \rangle$.

Let $\tilde{x}^Q_\theta$ denote the collective policy position on θ, and $\tilde{X}^Q$ denote the collective policy trajectory under parliament $\langle I, \vec{y}, \Phi^I \rangle$. On any given issue θ, the median policy positions in parliament are given by Definition 2. Since M is finite, there are at most two median positions, $x^{med}_{\theta-}$ and $x^{med}_{\theta+}$. These policy positions are Condorcet winners, in the sense that less than half of the political agents strictly prefer lower positions, and less than half of the political agents strictly prefer higher positions. In line with our earlier definitions, let x^{med}_θ denote the midpoint of these medians; $x^{med}_\theta = \frac{1}{2}\left[x^{med}_{\theta-} + x^{med}_{\theta+}\right]$. The medians $x^{med}_{\theta-}$ and $x^{med}_{\theta+}$ may well coincide, and they do so when M is odd. Then x^{med}_θ is the unique Condorcet winner. In general, we assign, on any issue θ, the midpoint median as the collective policy position of parliament $\langle I, \vec{y}, \Phi^I \rangle$ under majority rule, $\tilde{x}^Q_\theta = x^{med}_\theta$. The collective policy trajectory under parliamentary rule, $\tilde{X}^Q$, is given according to (11). As will be seen below, such a parliamentary policy trajectory is a compact set consisting of segments of the projection spheres of the platforms y^i of the parties $i \in I$ in parliament.

Example 4. *Consider again Figure 1. If the three ideologies marked in the diagram are held by political agents with equal weights, then the policy position of the agent A, located at* $y^1 = (-0.6, -0.2)$*, is their collective position on issue* L_θ*. If, by contrast, agent B, located at* $y^2 = (-0.4, 0.6)$*, has weight 2, and each of the other two has*

[23] Alternatively, the tie can be broken by a coin flip. The choice depends on how the modeler expects an impasse to be resolved. If there might be other factors outside of the model that would result in one of the positions prevailing, then breaking the tie with a coin flip is more suitable. If, on the other hand, the impasse is expected to be resolved through compromise, the current approach is more suitable.

[24] In the terminology of Durocher, Leblanc and Skala (2016), and when the mandates are positive integers, this defines a *multiset* I, with multiplicity m_i for each $i \in I$.

weight 1, then both $p_\theta(y^1) \approx (-0.36, 0.21)$ *and* $p_\theta(y^2) \approx (-0.56, 0.33)$ *are Condorcet winners, in which case their collective position is* $\frac{1}{2}\left[p_\theta(y^1) + p_\theta(y^2)\right] \approx (-0.46, 0.27)$.

Just as delegation to a single policy-maker incurs a nonnegative welfare loss for a given society, as compared with direct democracy, so does delegation to a parliament. Using the same specification as in the case of a single decision-maker, the cost of delegation to a parliament under majority rule is

$$K(Q) = \lambda(\theta) \int_\Theta \left| t_\theta\left(\tilde{x}_\theta^D\right) - t_\theta\left(\tilde{x}_\theta^Q\right)\right|^2 d\mu(\theta) \tag{22}$$

where μ is the issue distribution, and $\tilde{X}^D$ and $\tilde{X}^Q$ are the collective policy trajectories of the society under direct democracy (D) and under parliamentary democracy (Q), respectively. Clearly this cost of delegation under parliamentary democracy is non-negative, and it is zero if and only if the parliament's policy trajectory, $\tilde{X}^Q$, coincides with that of the electorate, $\tilde{X}^D$:

$$K(Q) \geq 0 \text{ with equality iff } \tilde{X}^Q = \tilde{X}^D \tag{23}$$

Remark 4. *Just as under autocracy and presidential democracy, citizens may attach utility to fixed traits of the parties in parliament. A welfare measure that includes this is*

$$\sum_{i \in I} \psi_i \mathbb{E}\left[\zeta_i\right] - K(Q)$$

where ψ_i *is the power of party* i *in parliament, the probability that it is pivotal on an issue when policy is decided in parliament, see next subsection.*

5.1. Political power in parliaments under majority rule.

For any issue $\theta \in \Theta$, let $\tilde{I}_\theta \subseteq I$ be the nonempty subset of political parties whose policy positions are Condorcet winners on θ. Any political party $i \in \tilde{I}_\theta$ is defined as *influential* on issue θ, and *pivotal* on issue θ if $\tilde{I}_\theta = \{i\}$. Being pivotal, its ideal policy will be implemented. Being influential but non-pivotal, a compromise policy, relatively close to its ideal policy, will be implemented.

Slightly generalizing Shapley (1977), we define the *political power* of a party $i \in I$ as the sum of (a) the probability of being pivotal when an issue arises, and (b) a term that represents situations in which it is non-pivotal but influential. To make this precise, suppose that issues $\theta \in \Theta$ are randomly drawn according to some probability measure μ on Θ, and let $\left|\tilde{I}_\theta\right| \in \mathbb{N}$ denote the number of influential parties on issue θ.

Definition 4. *The* ***political power*** *of a party $i \in I$ in parliament is*

$$\psi_i \;=\; \textstyle\sum_{n=1}^{|I|} \frac{1}{n} \Pr\left[\left|\tilde{I}_\theta\right| = n \text{ and } p_\theta(y^i) \in \left\{x_{\theta-}^{med}, x_{\theta+}^{med}\right\}\right] \tag{24}$$

We note that if the issue distribution μ is absolutely continuous and thus can be represented by a PDF on Θ, and the party ideologies are distinct, then the probability that two or more parties will project to the same point on any straight line L_θ is zero. In other words, $\left|\tilde{I}_\theta\right| \leq 2$ with probability one. Thus, the weighted sum of probabilities in (24) then equals the probability that party i will be pivotal, plus half the probability that it will be one of the two non-pivotal but influential parties:

$$\begin{aligned} \psi_i \quad = \quad & \Pr\left[\left|\tilde{I}_\theta\right| = 1 \text{ and } p_\theta(y^i) = x_\theta^{med}\right] \\ & + \frac{1}{2}\Pr\left[\left|\tilde{I}_\theta\right| = 2 \text{ and } p_\theta(y^i) \in \left\{x_{\theta-}^{med}, x_{\theta+}^{med}\right\}\right] \end{aligned} \tag{25}$$

The definition of power can be further simplified in generic situations, where not only all parties have distinct ideologies, as here assumed, but when, moreover, the mandates are such that no subset of political parties controls exactly half the mandates. This is the case with integer mandates m_i when their sum M is odd, as in any parliament with an odd number of seats. More precisely, suppose that the issue distribution μ is absolutely continuous, and that $\sum_{j \in J} m_j \neq M/2$ for all subsets $J \subset I$. Then $\left|\tilde{I}_\theta\right| = 1$ for almost every issue $\theta \in \Theta$, and thus

$$\psi_i = \Pr\left[t_\theta(y^i) = \tilde{t}_\theta\left(I\right)\right] \quad \forall i \in I \tag{26}$$

Example 5. *Consider a parliament with three political parties with equal mandates, and committed to ideologies as indicated in the diagram below, the same as given in Figure 1. The policy trajectories of the parties are the dashed circles, and the policy trajectory of the parliament (under majority rule) is the union of the solid curves. For each party the sector of issues where it is pivotal is the part of its policy trajectory marked as a solid curve. Under uniformly distributed issues, the power of each party is proportional to the size of its pivotal sector. For this example, the power of party A, B, and C are approximately 0.282, 0.523, 0.195 respectively. Thus, although all three parties have equally many mandates, party B has roughly twice as much political power as any other candidate. It is pivotal on double as many issues as any one of the other two parties. It is not hard to show that this result is true even if the three parties do not have exactly the same number of mandates. All that is needed is that no party single-handedly has a majority. To see this, note that for any issue, only three different permutations of the policy ordering of the parties are* *possible. If no party holds a majority of the mandates, then the party that projects between the other two is pivotal.*

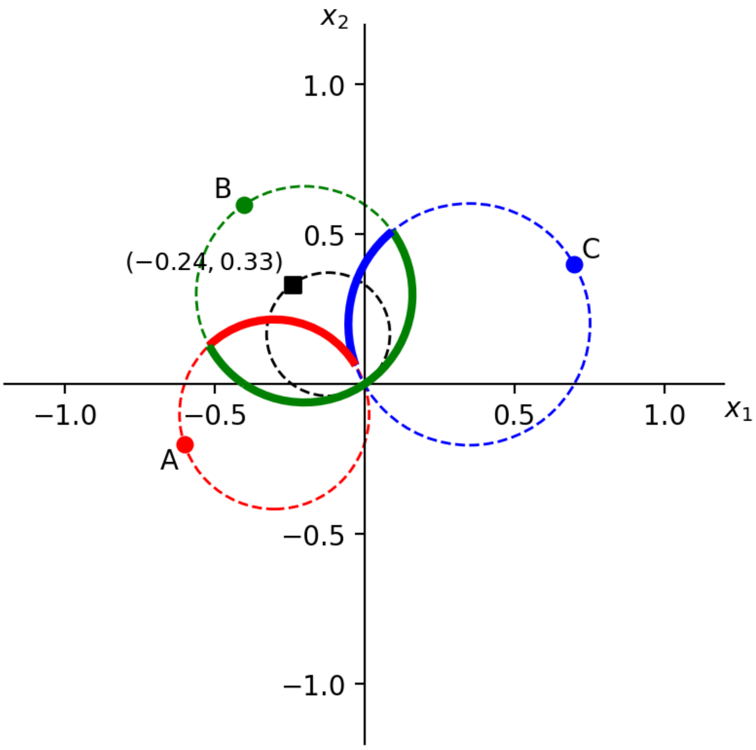

Figure 5: Projection trajectory of three parties with equal mandate, and their projection median.

5.2. Coalitions in parliaments. For political analysis, one often needs to consider coalitions of political parties. Consider, thus, any finite subset $C \subseteq I$ of political agents with platforms $y^i \in X$ and with positive mandates m_i. Let $\Phi^C : \mathbb{R}^d \to [0, 1]$ be the cumulative mass distribution function associated with this subset. The agents may be political parties in a parliament. Suppose that they wish to form a coalition, for example for the purpose of forming government. Two main approaches to such coalition formation appear natural. The *ex-post approach* is for the political parties to get together after an issue arises in parliament, and then apply majority rule within the coalition C to agree on a common policy to advocate in parliament on that issue. The second approach, the *ex-ante approach*, is that the parties in the coalition instead get together before any issue arises, and then pre-commit to a common ideology location, or *political platform*, $y^C \in X$, from which they will thereafter derive their common policy position on any issue that may arise.

The ex-post approach. For any issue $\theta \in \Theta$, let $y_\theta^C \in L_\theta$ denote the common position the members of coalition C agree on according to majority rule, after issue θ is realized. This position is given by the median of Φ^C in direction θ, as given in Definition 2.

The ex-ante approach. Under the ex-ante approach, what common platform will the parties in C commit to? We suggest they commit to that platform $y \in X$ which minimizes the weighted square distance from the platforms of the constituent parties, weighted by their political power before the coalition was formed. Since preferences are strictly convex, this uniquely determines the common location:

$$y^C = \arg\min_{x \in X} \sum_{i \in C} \psi_i \left\| x - y^i \right\|^2 \tag{27}$$

For parties with Euclidean preferences, y^C is a welfare-maximizing common ideology for the members of C, where each member is given not the weight that corresponds not to its number of seats in parliament but to its political power in parliament in the absence of the coalition. It is easily verified that this common ideology belongs to the convex hull of the platforms of the constituent parties, and, more exactly, is identical with the "power center" of C:

$$y^C = \bar{y}^C \tag{28}$$

where

$$\bar{y}^C = \sum_{i \in C} \left(\frac{\psi_i}{\sum_{j \in I} \psi_j} \right) y^i \tag{29}$$

Is there any connection between the power center of a coalition and its projection median? As stated in Section (3), the projection median of a coalition C is a location in ideology space $X = \mathbb{R}^d$ such that the average of its policy positions, when $\theta \in \Theta$ is uniformly randomly drawn, is the same as the average of the collective policy positions of C. Hence, one may think of the projection median as a location that "represents" the set of political agents C as a single location in ideology space. In force of Theorem 1 in Durocher, Leblanc and Skala (2017), the power center of a party coalition is also its projection median when issues $\theta \in \Theta$ are uniformly randomly drawn:

Theorem 1 [Durocher, Leblanc and Skala, 2017]**.** *If the issue distribution is uniform, then the power center of a finite set C coincides with the projection median of C, for any given weights $(\psi_i)_{i \in C}$.*

In other words, for uniformly random issues the welfare maximizing common platform of a coalition C in the *ex-ante* approach is both its power center and its projection median.

5.3. Empowerment.

Clearly, the political power of an *ex-ante* coalition C in a parliament may well differ from the sum of the political powers of its constituent members before the coalition was formed. We define the empowerment of a coalition C in the same vein as Alonso-Meijide and Carreras (2011). However, while they base their definition on the classical Shapley value, our definition is based on the spatial Shapley value applied to the above representation of ex-ante coalition formation. Moreover, while their definition is expressed in absolute terms, we define empowerment in relative terms. To be more precise, let C be any nonempty proper subset of I, and let $I^* = \{c\} \cup \{i \in I : i \notin C\}$, where c is the coalition C now treated as a single political agent with platform y^C. Let $\psi_i \in [0,1]$ be the political power of party $i \in I$ before C is formed, let ψ_c^* be the political power of the new political agent c, the coalition C, and let ψ_i^* be the political power of non-member party $i \in I^* \setminus \{c\}$ after coalition C is formed. In line with Afsar and Weibull (2021):

Definition 5. *The* ***empowerment*** *of any coalition member $i \in C$ that is not a dummy player is*

$$\eta_i = \frac{1}{\psi_i}\left(\frac{\psi_i}{\sum_{j\in C}\psi_j}\psi_c^* - \psi_i\right) \tag{30}$$

In other words, the empowerment of a non-dummy coalition member is defined as the relative change of its political power from joining the coalition, as compared with its original political power, where a coalition member's power is proportionate to its original political power. Written in a more concise form,

$$\eta_i = \frac{\psi_c^*}{\sum_{j\in C}\psi_j} - 1 \tag{31}$$

it is evident that the empowerment is the same for all coalition members, and, moreover, that it is positive if and only if the coalition's total power, ψ_c^*, when viewed as a single political agent, exceeds the sum of the constituent members' original powers.[25]

6. Political competition

We now step back in time and consider the election of political agents. The political agents may be individuals who compete for one office (say, to become president or

[25] Incentive considerations concerning coalition formation, including considerations regarding what non-coalition players can and might be willing to do after the formation of a coalition, have been thoroughly discussed and analyzed in the literature on stable sets, pioneered by von Neumann and Morgenstern (1944). See e.g. Harsanyi (1974), Peleg (1980), Hart and Kurz (1983), Chwe (1994), Xue (1998), and Ray and Vohra (2015a, 2015b, 2019).

governor), or political parties who compete for seats in a legislative body, such as a parliament or committee. Let again $I = \{1, ..., n\}$ be the set of political agents.

Political competition between political agents with fixed ideologies x^i, who are free to commit to arbitrary political platforms y^i, can be defined in multiple relevant scenarios, depending on the agents' goal functions, which may be (i) their numbers of votes, (ii) their vote shares, (iii) their resulting political power, or (iv) the utility to the party of the resulting policy decisions that will be taken by the winning presidential candidate or the elected parliament. The first motive, vote maximization, is relevant when the framework is applied to oligopolistic competition between producers, when these choose (potentially multidimensional) product varieties that are equally costly to produce and are sold at fixed (but potentially different) prices. Arguably, vote maximization is less relevant in political competition.[26]

Under proportional representation of political parties in parliament, the vote shares are also their shares of mandates (neglecting rounding off to integer numbers of seats). Given the shares of mandates and party platforms, each party's political power can be calculated, as well as the collective policy position of that parliament on any issue. These policy positions may be compared with decisions taken under direct democracy, and alternative forms of representative democracy, in terms of welfare, represented by the expected square difference in policies, for any given issue distribution.

For ease of exposition, in this section we assume that every issue is *ex ante* equally likely, and that voters give the same utility weight to all issues:

$$\mu = \kappa \quad \text{and} \quad \lambda(\theta) = \bar{\lambda} > 0 \ \forall \theta \in \Theta \tag{32}$$

Hence, according to Proposition 2, voters' preference over platforms are Euclidean, namely that between two platforms a voter prefers, *ceteris paribus*, the one that is closer to the ideology of the voter, and can be represented by the utility function given in (15). Furthermore, as specified in Section 2.4, voters also have (additively separable) idiosyncratic preferences over candidates or political parties, preferences that do not depend on the platforms chosen but solely on the characteristics of the political agents, such as candidates' personality or political parties' records. Moreover, a voter's choice is now between $n+1$ alternatives, to vote for one of the political agents or to abstain. Let the associated platform profile be denoted $\vec{y} = (y^1, ..., y^n)$. Under assumption (32), voters with ideology $x \in X$ attach (random) utility

$$\mathbb{E}_\mu \left[\tilde{U}_\theta (y, i, x) \right] = \zeta_i - \frac{\bar{\lambda}}{2} \left\| x - y^i \right\|^2 \tag{33}$$

[26] Below we argue that also plurality maximization, in contests between two political agents, is less relevant.

to a vote for a political agent i with valence ζ_i who is committed to platform $y^i \in X$. Here the expectation is taken with respect to the issues that will arise, while the randomness represents heterogeneity in the electorate concerning attitudes to fixed traits (personality, record etc.) of political agent $i \in I$. It follows that an individual voter's utility associated with each of the $n+1$ choice alternatives that the voter faces is

$$U_j(\vec{y}, x) = \begin{cases} \zeta_j - \frac{\lambda}{2}\|x - y^i\|^2 & \text{for } j \in I = \{1, ..., n\} \\ \varepsilon & \text{for } j = n+1 \end{cases} \tag{34}$$

where $\zeta_1, ...\zeta_n$ are the valences of the political agents, and ε is the abstention utility. Each voter is assumed to choose an alternative with the highest utility.[27] We recall that while a voter's utility is a random variable from the viewpoint of the political agents and outside observers, each voter attach a deterministic valence to each political agent and a deterministic value to abstention. Hence, each individual voter's choice is deterministic, but the choices of voters, even with the same ideology, is probabilistic from the viewpoint of political agents and outside observers. We assume that all random variables are statistically independent, both for each voter and also across voters.

We model the strategic interactions between the competing political agents as a simultaneous-move game, in which each political agent has a "true" ideology x^i in X, and all agents have the same pure-strategy set, the (nonempty, closed and convex) support X of the voter distribution. A pure strategy for player i is a platform $y^i \in X$. In scenarios (i) and (ii) all players' payoff functions are continuous, but not necessarily quasi-concave in their own choice of platform y^i. In scenarios (ii) and (iii), the games are constant-sum games since the sum of vote shares is always one, and the sum of all participants' power is also always one. In scenario (iv), the definition of payoff functions, and the question of their continuity is more subtle.

We will focus on two cases, the first being when all valences ζ_i are deterministic, that is, when the electorate is homogeneous in their evaluation of political agents' valences. In the second case, the valences ζ_i are random variables, that is, the electorate is heterogeneous not only in terms of ideology but also in terms of political agents' valences.

6.1. Deterministic valences.

Consider scenarios (i) and (ii), that is, situations in which all political agents are either vote maximizers or vote-share maximizers. In this first subsection, we focus

[27]When there is more than one alternative with maximal utility for a voter, the voter chooses among them with equal probability.

on the case when all valences $\zeta_1, ..., \zeta_n$ are deterministic while ε is a random variable with CDF F_0 with full support ($F_0(z) > 0$ for all $z \in \mathbb{R}$). Let

$$U_i(\vec{y}, x) = \zeta_i - \frac{\bar{\lambda}}{2} \left\| x - y^i \right\|^2 \tag{35}$$

be the utility that a voter with ideology $x \in X$ obtains from voting on political agent $i \in I$ with platform y^i and valence ζ_i, as discussed in Section (2.4), and let

$$\hat{U}(\vec{y}, x) = \max_{i \in I} U_i(\vec{y}, x) \tag{36}$$

This is the utility that a voter with ideology x obtains from voting on the political agent that the voter find best in terms of platform and valence. The probability that a voter with ideology $x \in X$ will choose to not abstain is thus $\Pr\left[\varepsilon \leq \hat{U}(\vec{y}, x)\right] = F_0\left[\hat{U}(\vec{y}, x)\right]$. The (normalized) *number of votes* for political agent i is thus

$$v_i(\vec{y}) = \int_{X_i(\vec{y})} F_0\left[\hat{U}(\vec{y}, x)\right] \phi(x)\, dx \tag{37}$$

where

$$X_i(\vec{y}) = \left\{ x \in X : U_i(\vec{y}, x) = \hat{U}(\vec{y}, x) \right\} \tag{38}$$

is the "voter support region" for political agent i, the (potentially empty) subset of voter ideologies x that (weakly) prefer political agent i, with platform y^i and valence ζ_i, over all other political agents. The union of all sets $X_i(\vec{y})$ is X, and the overlap between any two distinct such sets has Lebesgue measure zero. Election turnout,

$$T(\vec{y}) = \sum_{i \in I} v_i(\vec{y}) \tag{39}$$

is a number in the open unit interval, since the probability is positive both for the event that the random abstention utility ε is lower than at least one of the deterministic voting utilities $U_i(\vec{y}, x)$ and for the event that it is higher than all such voting utilities.

The *vote share* for each candidate i,

$$s_i(\vec{y}) = \frac{v_i(\vec{y})}{T(\vec{y})} \tag{40}$$

is thus well defined, and these shares always sum to one, so competition for vote shares is a constant-sum game. However, these payoff functions have discontinuities. This is the case, for example, in unidimensional ideology space when two political candidates with the same valence chose the same political platform and this is not the median of the voter distribution. Nevertheless, pure-strategy equilibria do exist in some settings. The following result for ideology spaces of arbitrary dimension $d \in \mathbb{N}$ is in line with Proposition 2:

Proposition 3. *Assume (32), and consider competition between two vote-share maximizing political agents with the same valence, when ϕ is unimodal and centrally symmetric around some location $x^o \in X$. There exists a unique Nash equilibrium, and in this equilibrium both choose x^o as their platform.*

(Proof in Appendix.)

We note that this median-voter result does not hold for more than two political agent, since if they are $n > 2$ in number, and all take on x^o as their platform, they each obtains vote share $1/n$, while a small unilateral deviation in any direction gives the deviator a vote share close to $1/2$.

The following simple example illustrates the possibility that abstention by alienation may induce vote-share maximizing political agents to take up distinct political platforms in equilibrium if the voter distribution is centrally symmetric but not unimodal. In the example, ideology space is one-dimensional and abstention utility deterministic. See Fournier et al. (2022) for general results in a similar deterministic setting, for ϕ symmetric and unimodal.

Example 6. *Consider the voter distribution ϕ on the line segment $X = [0, 6]$ shown in the diagram below. This voter distribution is centrally symmetric around $x^o = 3$. Assume that voters abstain from voting if the nearest political agent is more than 1 distance unit away. Imagine three political parties, each striving to maximize its vote share. If they choose platforms 1, 3 and 5, respectively, they each obtain vote share $1/3$. No party can increase its vote share by moving away from its current platform.*[28] *This equilibrium is also unique, up to the permutation of the parties, and strict. The same reasoning holds if there were only two political parties. In equilibrium they locate at any two distinct platforms in the subset $\{1, 3, 5\}$.*

[28]By moving slightly towards a nearest competitor, the party loses double as many votes as it gains, because of abstention. If a party would move to the same platform as one of the other parties, its vote share would fall to $1/4$, and if party 1 or 3 would place itself exactly between its two competitors, its vote share would become $1/8$.

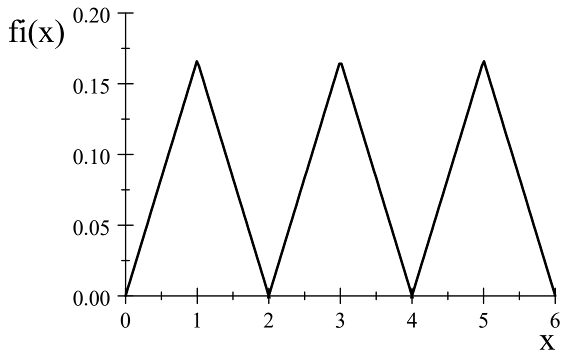

Figure 6: A non-unimodal but centrally symmetric voter distribution.

6.2. Random valences.

Consider the same setting as in the previous subsection, but now suppose that also the valence terms ζ_i of the competing political agents $i \in I$ are random variables. As mentioned above, the random variables ε, ζ_1, ..., ζ_n are taken to be statistically independent and to have probability distributions F_0, F_1, ..., F_n. Let $q_i(\vec{y}, x)$ be the probability that a voter with ideology $x \in X$ votes for political agent $i \in I$ if the agent is committed to platform y^i, let $v_i(\vec{y})$ be the number of votes on political agent $i \in I$, let $s_i(\vec{y})$ be the agent's vote share, and let $q_0(\vec{y}, x)$ be the associated abstention rate, that is, the probability that a voter with ideology $x \in X$ will not vote, given the agents' platform profile $\vec{y}$. The following result follows from a slight modification of the proof of Proposition 3:

Proposition 4. *Assume (32), and consider competition between two vote-share maximizing political agents with the same valence distribution, when ϕ is unimodal and centrally symmetric around some ideology $x^o \in X$. There exists a unique Nash equilibrium. In this equilibrium both agents choose x^o as their political platform.*

(Proof in Appendix.)

This generalized version of the median-voter theorem is fragile. It ceases to hold if any one of its hypotheses is violated. This is most easily demonstrated in the one-dimensional case. Suppose that, just as in Hotelling (1929), ϕ is uniform on $X = [0, 1]$, and let there be two vote-share maximizing political candidates, with platforms $y_1, y_2 \in [0, 1]$. If all voters always vote (zero abstention rate), then vote-share maximization is equivalent with vote maximization, and in that case both candidates will, in equilibrium, locate at the median, $y_1^* = y_2^* = 1/2$. The above proposition says that this is still true for vote-share maximizing candidates, even if voters may abstain from voting due to alienation. It is shown numerically in the next subsection that the claim in the proposition is not generally true for vote-maximizing candidates. The claim is also invalid if one keeps vote-share maximization as a goal for the competitors but relaxes the assumption of equal valence. Also this is shown numerically in the next subsection.

Example 6 shows that there are non-unimodal but symmetric voter distributions under which the claim in Proposition 4 does not hold. What about unimodal but asymmetric voter distributions? Again the claim may fail. Let ϕ be a skewed PDF on $X = [0, 1]$ with median below $1/2$. For example, $\phi(x) = 2(1 - x)$. If both candidates were to locate at the median, they would both obtain half the votes given. If voters never abstained, that would also be an equilibrium (indeed the unique equilibrium), in agreement with the median-voter theorem. However, if there is significant abstention due to alienation, then this is not an equilibrium, since any one of the two candidates, if they have the same valence, can obtain more than half the votes by moving slightly to the left of the median, the voter density being higher on that side of the median.

The assumption that there are only two competitors is also crucial for the validity of the claim in the proposition. With three or more competitors, it is not an equilibrium for them all to take the median location (under the otherwise maintained hypothesis of the proposition), since a slight move away would give the deviator a vote share close to $1/2$, instead of the vote share $1/n$ at the median location (for $n > 2$). If there is no abstention, no pure-strategy equilibrium exists. However, pure-strategy equilibria can exist when there is abstention, as seen in Example 6, equilibria in which the candidates espouse distinct political platforms.

Gumbel distributed valences. For general valence distributions F_i, the voting probabilities are obtained by conditioning and integration, which may result in complicated expressions. However, the Gumbel distribution belongs to a class of probability distributions for which "invariance of the maximum" holds (see Fosgerau et al., 2018), a property that in the present context implies that the probability for a vote on any given political agent is the product of the probability of participation (non-abstention) and the probability for a vote on the candidate if abstention had not been an option. More specifically, suppose that the valence ζ_i of each political agent $i \in I$ is *Gumbel*(ν_i, β) distributed, that is, the CDF of the random valence ζ_i is defined by[29]

$$F_i(t) = \Pr[\zeta_i \leq t] = \exp\left[-e^{-(t-\nu_i)/\beta}\right] \qquad \forall i \in I \tag{41}$$

for some $\nu_i \in \mathbb{R}$ and $\beta > 0$.

[29] Here ν_i is the mode, usually called the *location* parameter, and $\beta > 0$ is the *scale* parameter. The mean is $\nu_i + \gamma\beta$, where γ is Euler's constant, and the standard deviation is $\pi\beta/\sqrt{6}$.

Proposition 5. *Assume (32), and consider $n \in \mathbb{N}$ political agents committed to platforms $y^1, ..., y^n \in X = \mathbb{R}^d$ for some $n, d \in \mathbb{N}$. Let the abstention utility ε have CDF F_0 and assume that the political agents' valences ζ_i are independent Gumbel(ν_i, β) distributed random variables. Then*

$$q_0 (\vec{y}, x) = \int_{-\infty}^{+\infty} \exp \left(-e^{-s/\beta} \sum_{i \in I} e^{\left(\nu_i - \frac{\bar{\lambda}}{2} \left\| x - y^i \right\|^2\right)/\beta} \right) dF_0 (s) \tag{42}$$

$$q_i (\vec{y}, x) = (1 - q_0 (\vec{y}, x)) \cdot \frac{e^{\left(\nu_i - \frac{\bar{\lambda}}{2} \left\| x - y^i \right\|^2\right)/\beta}}{\sum_{j \in I} e^{\left(\nu_j - \frac{\bar{\lambda}}{2} \| x - y^j \|^2\right)/\beta}} \qquad \forall i \in I \tag{43}$$

(Proof in the Appendix.)

It may be noted that all choice probabilities in the proposition, $q_0 (\vec{y}, x)$, $q_1 (\vec{y}, x)$,..., $q_n (\vec{y}, x)$, are positive and differentiable functions of the agents' platforms, $\vec{y} = (y^1, ..., y^n)$. Thus also the number of votes,

$$v_i (\vec{y}) = \int_X q_i (\vec{y}, x) \phi (x) \, dx \quad \forall i \in I \tag{44}$$

and vote shares,

$$s_i (\vec{y}) = \frac{v_i (\vec{y})}{\sum_{j=1}^n v_j (\vec{y})} \quad \forall i \in I \tag{45}$$

are differentiable. Not surprisingly, the abstention rate $q_0 (\vec{y}, x)$ among voters at any ideology x is higher the further their ideology x is from the platform of any of the political agents, *ceteris paribus*. Likewise, the probability $q_i (\vec{y}, x)$ that a voter with any ideology x will vote for political agent i is decreasing in the distance between ideology x and the political platform y^i of that agent, ceteris paribus. This is seen in equations (42) and (43): an increase in $\|x - y^i\|$ increases $q_0 (\vec{y}, x)$ and decreases the second factor in the expression for $q_i (\vec{y}, x)$. The voter turnout in the election is

$$T (\vec{y}) = \sum_{i \in I} v_i (\vec{y}) = \int_X (1 - q_0 (\vec{y}, x)) \phi (x) \, dx \tag{46}$$

The formulae in Proposition 5 are further simplified if also the abstention utility is Gumbel distributed:

Corollary 2. *If the CDF F_0 in Proposition 5 is the Gumbel(ν_0, β) distribution, then*

$$q_i (\vec{y}, x) = \frac{e^{\left(\nu_i - \frac{\bar{\lambda}}{2} \left\| x - y^i \right\|^2\right)/\beta}}{e^{\nu_0/\beta} + \sum_{j \in I} e^{\left(\nu_j - \frac{\bar{\lambda}}{2} \| x - y^j \|^2\right)/\beta}} \quad \forall i \in I \tag{47}$$

and

$$q_0(\vec{y}, x) = \frac{e^{\nu_0/\beta}}{e^{\nu_0/\beta} + \sum_{j \in I} e^{\left(\nu_j - \frac{\lambda}{2}\|x - y^j\|^2\right)/\beta}} \tag{48}$$

It follows from (47) that the probability that voters at an ideology x will vote for a certain political agent $i \in I$ is an increasing function of the Euclidean distance in ideology space from ideology x to the platform y^j of any *other* political candidate or party $j \in I$. Formally:

$$\frac{dq_i(\vec{y}, x)}{d(\|x - y^j\|)} > 0 \quad \forall i, j \in I, \ i \neq j \tag{49}$$

6.3. Policy-motivated political agents.

We here briefly elaborate on scenario (iv) for the case of elections to parliaments with proportional representation, and with majority rule within the parliament.[30] Let thus $I = \{1, ..., n\}$ be the set of political parties with "true" ideologies $x^i \in X$ competing for seats in the parliament. Let their chosen policy platforms in the election be given by the vector $\vec{y} = (y^1, ..., y^n)$. The electorate is defined by the voter distribution ϕ over ideologies, along with the CDFs F_0, F_1, ... , F_n. For a given electorate and platform profile $\vec{y}$, let $\vec{s} = (s_1, ..., s_n)$ be the resulting vote shares. Under proportional representation, and neglecting the rounding off to whole numbers, the mandate shares m_i/M are equal to the vote shares s_i. Thus $\vec{m} = (m_1, ..., m_n)$ is proportional to $\vec{s}$ (and, more generally, $\vec{m}$ is a function of $\vec{s}$). After the parliament has been elected, issues $\theta \in \Theta$ arise. They are randomly drawn according to some exogenous and absolutely continuous probability measure μ on Θ.

Consider the simultaneous-move game between the n political parties, where, as mentioned above, each party's strategy is a choice of platform, $y^i \in X$, and its goal is to minimize the expected squared distance between the majoritarian policy in parliament, as defined in Section 5, and its own ideal policy, on a randomly drawn issue, from the vantage point of the party's "true" ideology location $x^i \in X$. For any issue $\theta \in \Theta$, let $\tilde{x}^I_\theta \in \mathbb{R}$ be the parliamentary policy decision under majority rule. By its choice of political platform $y^i \in X$, each party i strives to minimize the expected squared policy distance

$$D_i(\vec{y}) = \int_{\theta \in \Theta} \left| t_\theta\left(\tilde{x}^I_\theta\right) - t_\theta(x^i)\right|^2 d\mu(\theta) \tag{50}$$

[30] In the case of elections to an office, a candidate will influence the subsequent policy outcomes only by winning the election. Hence, if the election is made in one round, a candidate can have a policy influence only by obtaining the largest voter share among all candidates. Hence, vote share maximization is then a good proxy. In a two-round election, the choice of platform is more subtle.

This defines for each party $i \in I$ its loss function $D_i : X^n \to \mathbb{R}_+$, that it strives to minimize by way of choosing its own platform y^i. Hence, $-D_i$ is the payoff function of player $i \in I$ in this game.

Such payoff functions are more complex than maximization of vote shares. In particular, unlike votes and vote shares, they exhibit discontinuities. To see this discontinuity in the simplest setting, consider competition in one-dimensional ideology space between two purely policy-motivated political agents with the same valence distribution, when ϕ is unimodal and centrally symmetric around some ideology x^o. Suppose that the agents' true ideologies, x^1 and x^2, are on different sides of the voter median x^o, say $x^1 < x^o < x^2$. Consider any two political platforms y^1 and y^2 adopted by the parties, where $y^1 \leq x^o \leq y^2$, that is, the left-wing agent chooses a platform that is not to the right of the median voter, and likewise for the right-wing agent. If agent 1 positions itself closer to the median voter than agent 2, $|y^1 - x^o| < |y^2 - x^o|$, then agent 1 wins a majority of the votes given, by unimodality and symmetry of the voter distribution ϕ, so then the implemented policy, after the election, will be according to 1's platform, that is, $y^* = y^1$. Likewise if $|y^1 - x^o| > |y^2 - x^o|$; then 2 wins and $y^* = y^2$. Thus, there is a discontinuity in 1's payoff function whenever $|y^1 - x^o| = |y^2 - x^o|$, since then the electorate is split 50/50. See the diagram below, drawn for $x^o = 0$ and $y_2 = 1$. The discontinuity is not due to any discontinuity in agents' preferences *per se* but by the discontinuity of majority rule as a function of vote shares.

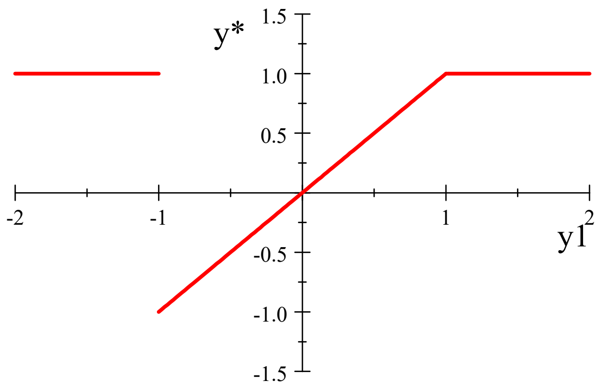

Figure 7: A discontinuity in a policy-motivated agent's goal function.

Despite such discontinuities in payoff functions, pure-strategy Nash equilibria do exist in some special cases also for policy-motivated political agents. The proof of the following result is a slight extension of the proof of Proposition 4. In essence, it establishes that if all issues are equally likely, then also competition between two policy-motivated political agents, under otherwise the same conditions as in that proposition, will in equilibrium generically result in the same outcome; the median voter's ideal policy will be implemented for every issue that may arise after the election. The non-generic exception is when the political agents' ideologies differ from the median voter's location in exactly the same direction. Then the agent whose ideology is closest to that of the median voter wins. Let y^* denote the winning political platform.

Proposition 6. *Assume (32), and consider competition between two purely policy-motivated political agents with the same valence, when ϕ is unimodal and centrally symmetric around some ideology $x^o \in X$. Suppose that the ideologies x^o, x^1 and x^2 are distinct. There then exist Nash equilibria $\vec{y} = (y^1, y^2)$ in which $y^i = x^o$ for at least one agent i. In all these equilibria, $y^* = x^o$. If $x^1 - x^o = \delta (x^2 - x^o)$ for some $\delta > 0$, then there also exists a continuum of Nash equilibria, all with the same outcome $y^* = \arg\min_{x \in \{x^1, x^2\}} \|x - x^o\|$.*

The claims in the proposition presume that x^o, x^1 and x^2 are distinct. However, it is easily verified that if one of the two agents' ideologies would coincide with that of the median voter, then that agent will adopt x^o as its platform in equilibrium, and thus again $y^* = x^o$. So the first claim in the proposition holds also in this case. It is also easily verified that the second claim in the proposition holds also if the two agents would happen to have exactly the same ideology. We also note that there is a whole continuum of equilibria that give rise to each of the two outcomes. However, if the political agents, aside from their policy motivation, which is here assumed to be their only concern, would care the slightest also about their vote share, then the associated equilibria would be unique, for each of the two possible outcomes.

6.4. Plurality-maximizing political agents.

Much of the literature on spatial political competition focuses on the mathematically more tractable case of two competing political agents, each of whom either strives to maximize its total number of votes, or its plurality, where plurality is defined as the difference between its number of votes and the number of votes for the competitor.[31] In the latter case, the goal function of candidate i is $v_i(\vec{y}) - v_j(\vec{y})$, for

[31] A third objective function used in the voting literature is the maximization of one's winning probability. However, here the electorate is modeled as a continuum, so randomness is washed away. See Laslier (2005) for a discussion of how maximization of the probability of winning relates to plurality maximization.

$i = 1, 2$ and $j \neq i$. In general, the competitors' best replies are qualitatively different from those under vote maximization and those under vote-share maximization.[32] Situations in which the set of equilibria for vote maximizers differs from that for plurality maximizers are given in Hinich and Ordeshook (1970), and Aranson, Hinich, and Ordeshook (1974).

However, in some special circumstances, the equilibria between plurality maximizing competitors coincide with the equilibria between vote-share maximizing competitors. To see this possibility in the case of random valence and abstention utility, as described in section 6.2, consider the necessary first-order conditions for equilibrium in unidimensional ideology space. In this special case we simplify notation and write y_1 and y_2 for the two candidates' platforms. For plurality maximizing candidates, the necessary first-order conditions are

$$\frac{\partial v_i(\vec{y})}{\partial y_i} = \frac{\partial v_j(\vec{y})}{\partial y_i} \quad \text{for } i = 1, 2 \text{ and } j \neq i \tag{51}$$

while for vote-share maximizing candidates they are

$$v_j(\vec{y}) \frac{\partial v_i(\vec{y})}{\partial y_i} = v_i(\vec{y}) \frac{v_j(\vec{y})}{\partial y_i} \quad \text{for } i = 1, 2 \text{ and } j \neq i \tag{52}$$

These two equation systems coincide if $v_2(\vec{y}) = v_1(\vec{y})$. This is the case when the candidates have the same valence ($\zeta_1 = \zeta_2$), and symmetrically located around the mode of a unimodal and symmetric voter distribution. Thus, the first-order conditions for vote-share maximizing candidates may coincide with the first-order conditions for plurality maximizing candidates. However, in more general situations, the F.O.C.:s, and hence equilibria, may differ in contests between plurality maximizers and vote-share maximizers. Arguably, both in elections to parliaments with proportional representation, and in presidential elections, the latter goal is more natural.

The qualitative differences between "vote maximization", "vote share maximization" and "plurality maximization" are illustrated in the three diagrams below, where each diagram shows isoquants for one of the goal functions. On the horizontal axis is the number of votes on agent 1, and on the vertical the number of votes on agent 2. With an electorate normalized to unit size, these two numbers are nonnegative and their sum does not exceed unity.

[32] It may be noted that vote-share maximization for a political agent i is equivalent with maximization of its vote ratio:

$$\arg\max_{y^i} \frac{v_i(\vec{y})}{v_i(\vec{y}) + v_j(\vec{y})} = \arg\max_{y^i} \frac{v_i(\vec{y})}{v_j(\vec{y})}$$

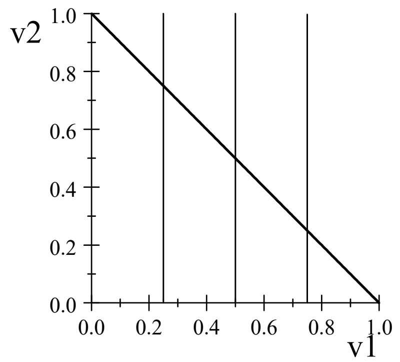

Figure 8 (a): Isoquants for a vote maximizing candidate 1.

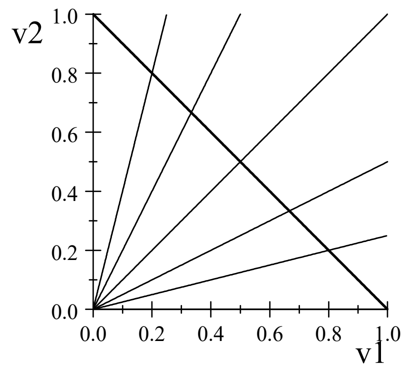

Figure 8 (b): Isoquants for a vote-share maximizing candidate.

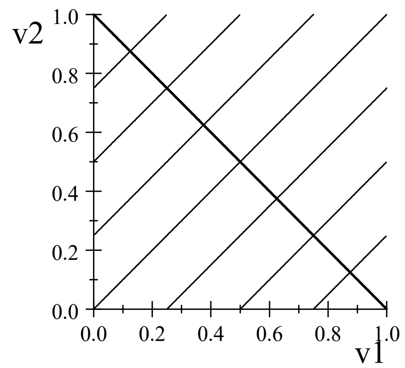

Figure 8 (c): Isoquants for a plurality maximizing candidate.

Now consider an ideology space of arbitrary finite location, again with two competing political agents. Suppose, first, that the two political agents have the same valence and that the voter distribution ϕ is such that about 60% of the electorate is located near one location, say $x_A \in X$, and 40% near another location, $x_B \in X$, at some distance from each other. Suppose that voters abstain unless the nearest agent is quite close to themselves (as compared with the distance $\|x_A - x_B\|$ between the two locations). Suppose that agent 1 has located at x_A. Then a vote-maximizing agent 2 would locate at (or near) x_B, thereby obtaining about 40% of the electorate's total number of votes, rather than about 30% if locating at (or near) x_A. By contrast, a vote-share maximizing agent 2 would locate at x_A, next to agent 1, thus obtaining about half the votes given, rather than approximately the vote share 0.4 if locating at x_B. Also a plurality-maximizing agent 2 would locate at x_A, thus obtaining plurality zero, instead of -20% at x_B. Unlike competition for votes, competition for vote shares or plurality may thus lead to platform convergence, leaving a large fraction of voters alienated and non-participating in the election.

In this toy example, vote-share maximization led to the same location as plurality maximization. But this is not always the case. To illustrate this possibility, take the same voter distribution ϕ as above, but now suppose that agent 1 has higher valence than agent 2 and obtains 65% of the votes given if both agents are at the same location in ideology space. Suppose that agent 1 has located at x_A. A plurality-maximizing agent 2 would then still locate at x_A, thereby obtaining plurality -18%, instead of -20% if locating at x_B. However, a vote-share maximizing agent would choose location x_B, thus obtaining vote share 0.4, instead of 0.21 at x_A.

The final subsection examines scenarios (i) and (ii) by means of numerical examples.

6.5. Numerical examples.

We here analyze the best-reply correspondences in competition between two political agents, who may be candidates for presidency or political parties competing for seats in parliament under proportional representation, in unidimensional ideology space, both under vote maximization and under vote-share maximization. Thus $d = 1$ and $n = 2$. We compare two societies with different voter distributions. In one society, voters' ideologies are uniformly distributed on the interval $[-1, 1]$. In the other society, the voter distribution is normal, with the same mean and variance as in the first society (that is, in both societies the voter distribution has mean value zero and variance 1/3). See the diagram below. In these simulations, the scale parameter $\bar{\lambda}$ was normalized to 2, and the abstention utility was deterministic and normalized to zero. By assuming deterministic abstention utility, the numerical results become sharper.

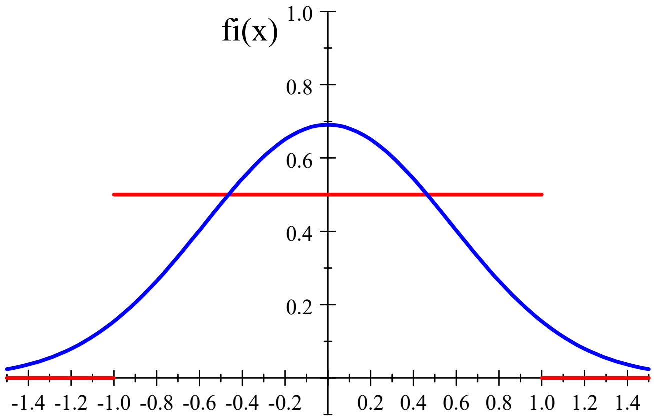

Figure 9: The two voter distributions in the numerical examples.

For each society, we consider four cases, namely, when the valence ζ_i of each agent $i = 1, 2$ is $Gumbel(\nu_i, \beta)$ for $\beta = 0.2$**,** and

> Case I: the agents have the same valence distribution, $\nu_1 = \nu_2 = 0.5$, and each agent strives to maximize its (a) votes, and (b) vote share,
>
> Case II: The agents have different valence distributions**,** $\nu_1 = 1$ and $\nu_2 = 0.5$, and each agent strives to maximize its (a) votes, and (b) vote share.

We first analyze the society with a uniform voter distribution, since this is closest to the familiar setting in Hotelling (1929).

Uniform voter distribution case I. The diagram below shows the two best-reply correspondences in case I (a) for the society with a uniform voter distribution. The solid (blue) curve is the best-reply correspondence of agent 1 and the dashed (red) curve that of agent 2. When the location of agent 2 is precisely at zero, agent 1 has two best replies, one at approximately -0.18 and one at $+0.18$.

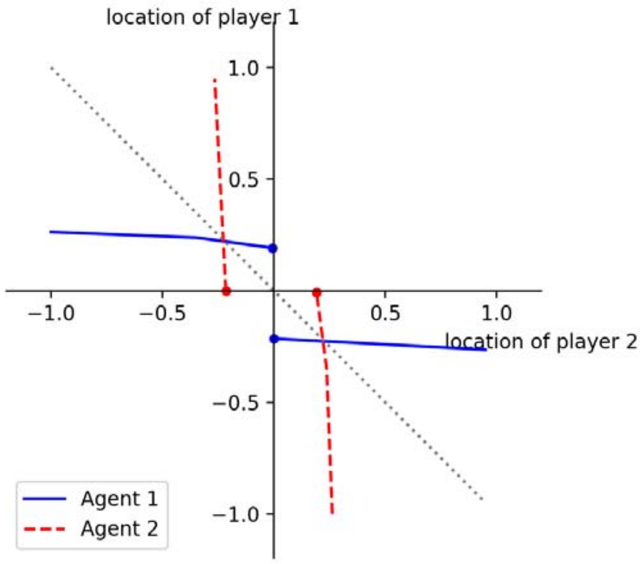

Figure 10 (a): Best-reply correspondences for equally popular vote maximizing agents, uniform voter distribution.

We note that best reply of each agent lies on the opposite side of the opponent's strategy with respect to the median voter. The diagram suggests two symmetric equilibria, in each of which one agent is located at approximately -0.2 and the other at approximately $+0.2$. The present model does thus not predict that the two agents converge to the median voter. Instead, they stand at some distance from each other. The reason is abstention. If all voters would always cast a vote, then we know, since Hotelling (1928), that the unique Nash equilibrium would be for both agents to locate at zero, the median voter's ideology.

The diagram below shows the best-reply graphs for two agents with the same valence, now maximizing their vote shares. Again the solid curve shows the best replies for agent 1 and the dashed curve the best replies for agent 2. The diagram suggests a unique equilibrium, namely, convergence of both agents to the median voter ideology. The repelling force (because of abstention) that we saw act in Case I (a) is now counter-acted by an attracting force, namely the interest to reduce the number of votes for the competitor and thereby enhance one's own vote share. We also note that, in contrast with the previous case, the best reply of an agent lies on the same side as the opponent's location, between the opponent and the median voter.

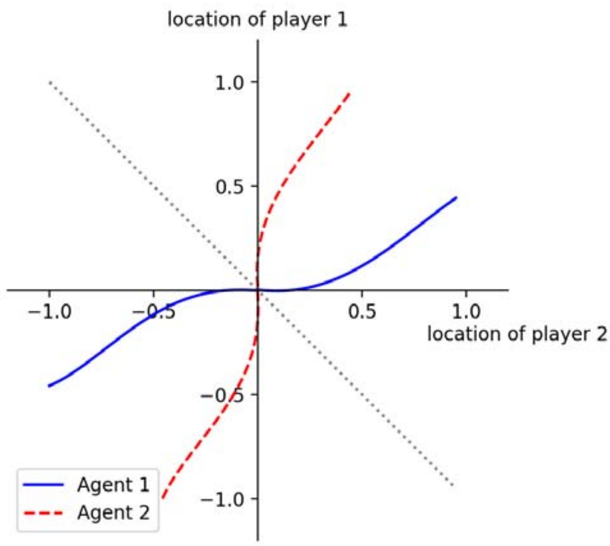

Figure 10 (b): Best-reply correspondences for vote–share maximizing agents with equal valence, uniform voter distribution.

Uniform voter distribution case II. Here we consider the case when agent 1 has higher valence than agent 2, and when each agent strives to maximize her number of votes. In the diagram below, the dashed best-reply curve for agent 2 jumps from about +0.6 when agent 1 is located just to the left of zero, to about −0.6 when agent 1 is located just to the right of zero. When agent 1 is located precisely at zero, agent 2 has two best replies, at approximately ±0.6. We note that now the (high valence) agent 1 wants to be on the same side as the (low valence) agent 2, while agent 2 prefers to locate on the opposite side with respect to the electorate median.

The diagram suggests that there exists no pure-strategy equilibrium, since the two best-reply correspondences do not intersect. However, we conjecture that there exists a mixed equilibrium in which agent 1 locates at the median voter, while agent 2 (with lower valence) locates at either (approximately) −0.6 or +0.6, with equal probability for both. (Had agent 1 known the realization of 2's mixed strategy, then 1 would have moved away from the median voter, slightly towards the low-valence agent.)

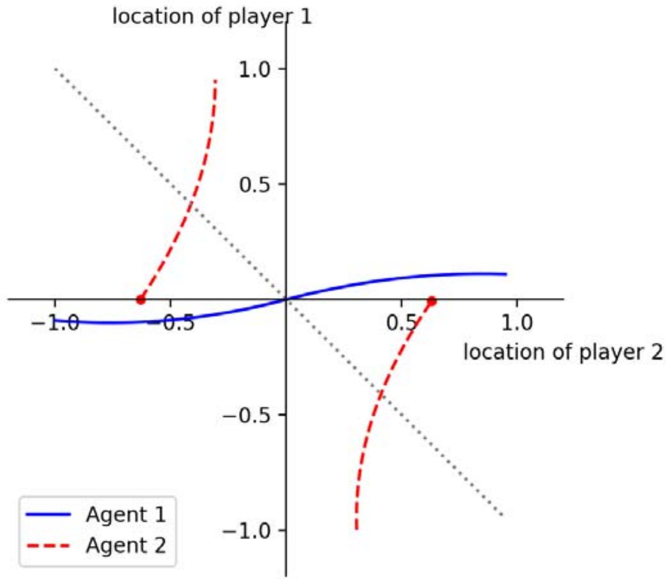

Figure 11 (a): Best-reply correspondences for vote maximizing agents with unequal valence, uniform voter distribution.

Below is the diagram again for two agents with unequal valence, but this time each agent strives to maximize her vote share. The horizontal segment of the dashed (red) curve is a numerical artefact. When player 1 is located precisely at the center, player 2 has two best replies, at approximately −0.6 and +0.6. The diagram again suggests that there exists no pure-strategy equilibrium, and that there exist a mixed equilibrium in which agent 1 locates at the median voter, while agent 2 (with lower valence) locates at either (approximately) −0.6 or +0.6, with equal probability for both. While the mixed equilibrium resembles the one in the previous case, we note that now the incentive for the high valence agent to match the location of the low valence agent is much stronger, in the sense that the agent's best reply curve is closer to the diagonal.

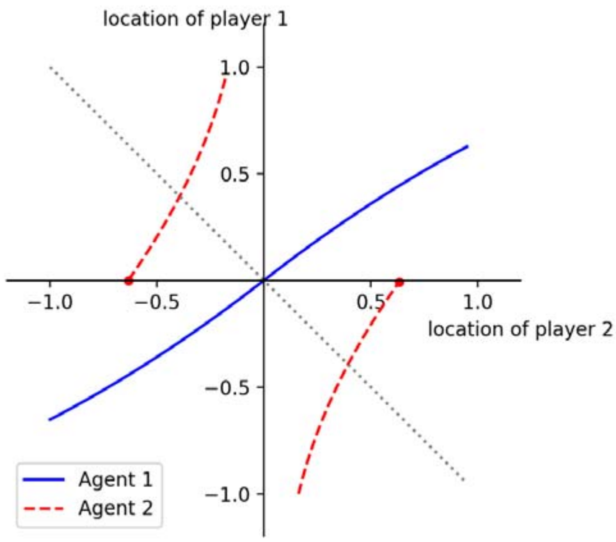

Figure 11 (b): Best-reply correspondences for vote–share maximizing agents with unequal valence, uniform voter distribution.

Next we consider the same four cases in society B, where the voter distribution is the standard normal distribution.

Normal voter distribution case I. We begin the study of the society with normally distributed voters by considering two agents with the same valence, each aiming to maximize its numbers of votes. Although the variance of the voter distribution is the same as in the uniform case, the agents now converge to the median voter in the unique equilibrium, see Figure 12 (a). Figure 12 (b) shows the best-reply diagram when the two agents, still with equal valence, instead strive to maximize their vote shares. Just as in the case of a uniform voter distribution, the unique equilibrium is that both agents locate at the median voter.

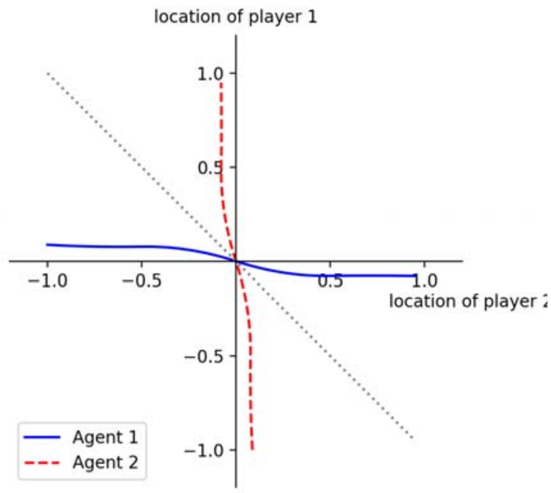

Figure 12 (a): Best-reply correspondences for vote maximizing agents with equal valence, normal voter distribution.

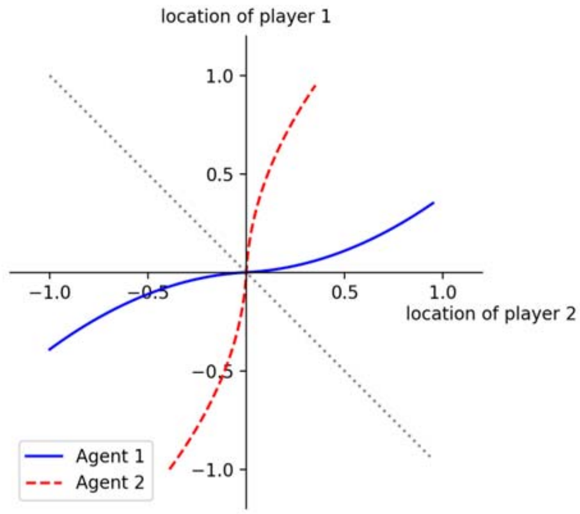

Figure 12 (b): Best-reply correspondences for equally popular vote–share maximizing agents, normal voter distribution.

Normal voter distribution case II. Next is two agents with different valence striving to maximize their numbers of votes. The best reply of agent 2, who has lower valence, is to locate at the other side of the median voter from the agent with higher valence. Figure 13 (a) suggests that there exists no pure-strategy equilibrium. However, it also suggests a mixed equilibrium in which agent 1 locates at the median voter and agent 2 locates at either (approximately) -0.65 or $+0.65$, with equal probability for both.

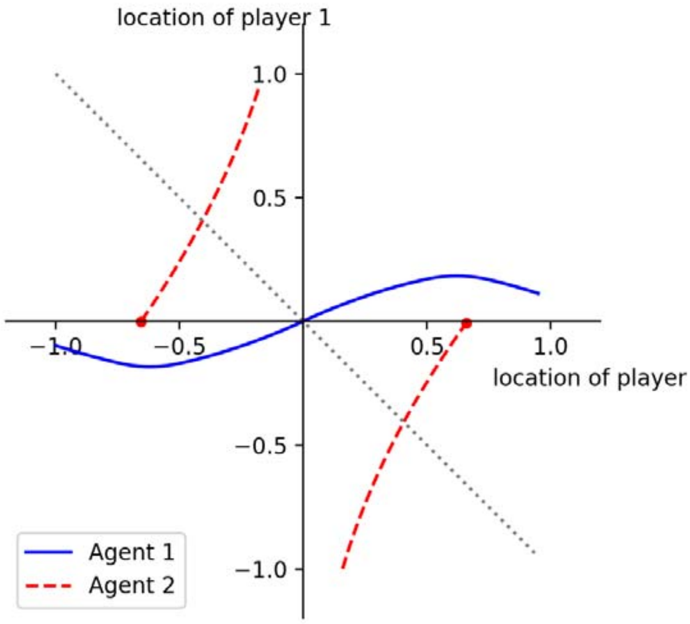

Figure 13 (a): Best-reply correspondences for unequally popular vote maximizing agents, normal voter distribution.

Finally, we consider the case when two agents with differing valence each strive to maximize its vote share. We note that player 1, who's valence is higher, has a virtually linear best-response curve with slope approximately 0.7. Hence, this player wants to locate itself on the same side as player 2, but closer to the median voter, at about 70% of 2's distance from the median voter. As in the case under uniform distribution, the diagram suggests that there is no pure equilibrium also in this case. When the high-valence agent 1 is located precisely at the median voter, then it appears that agent 2 has two best replies, at approximately ± 0.6. We conjecture that this game has a unique mixed equilibrium, in which the contestant 1 locates at the median voter and the contestant 2 assigns probability one half to each of its best pure replies, -0.6 and $+0.6$. (Had agent 1 known the realization of this randomization, it would have moved away from the median voter toward agent 2, and moved more than halfway towards that agent.)

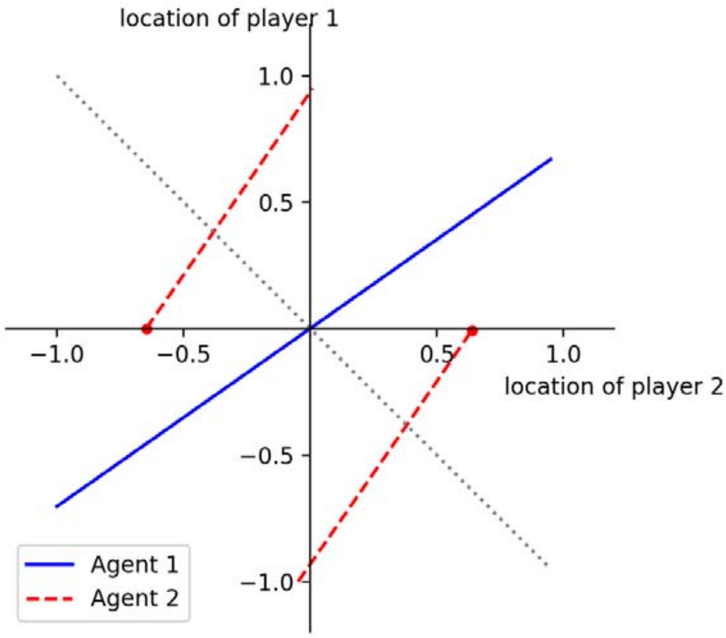

Figure 13 (b): Best-reply correspondences for unequally popular vote–share maximizing agents, normal voter distribution.

7. Application to the Swedish parliament

In the Swedish general election in September 2022, eight political parties passed the 4% threshold for representation in the parliament, the "Riksdag". The distribution of the 349 mandates in the 2022 Riksdag is shown in the table below.

Table 1: Party composition of the 2022 Swedish Riksdag.

party	seats	share
S (Socialdemokraterna)	107	0.307
SD (Sverigedemokraterna)	73	0.209
M (Moderaterna)	68	0.195
C (Centerpartiet)	24	0.069
V (Vänsterpartiet)	24	0.069
KD (Kristdemokraterna)	19	0.054
MP (Miljöpartiet)	18	0.051
L (Liberalerna)	16	0.046
	349	1

The next diagram, based on data from Chapel Hill's expert survey (Jolly et al., 2019), shows the platforms of all parties.

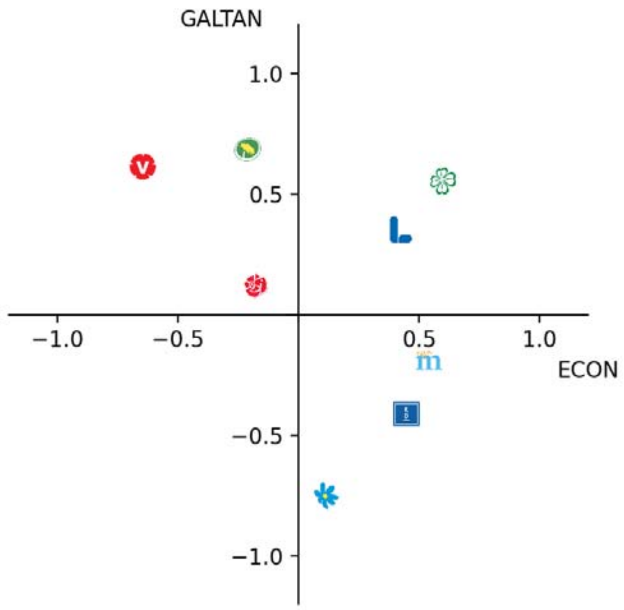

Figure 15: GAL-TAN diagram over the eight political parties in the Swedish Riksdag.

Here S is marked by a rose, MP by a yellow flower against a green background, C by cloves, and SD by a blue flower. Both the ECON and the GAL-TAN scale (which usually ranges from zero to ten) has been normalized to the interval between -1 and +1.[33]

The next diagram shows the same GAL-TAN diagram, but now as one colored dot for each party. The diagram also depicts the policy trajectory of each party, namely the policy positions on each issue of each party. The flower-shaped set in the middle is the collective policy trajectory of the parliament. It shows the collective policy positions on all issues, as defined in Section 5, and the colors indicate which party is then pivotal on that issue. The collective policy trajectory is composed of segments of the pivotal parties' policy trajectories, here circles.

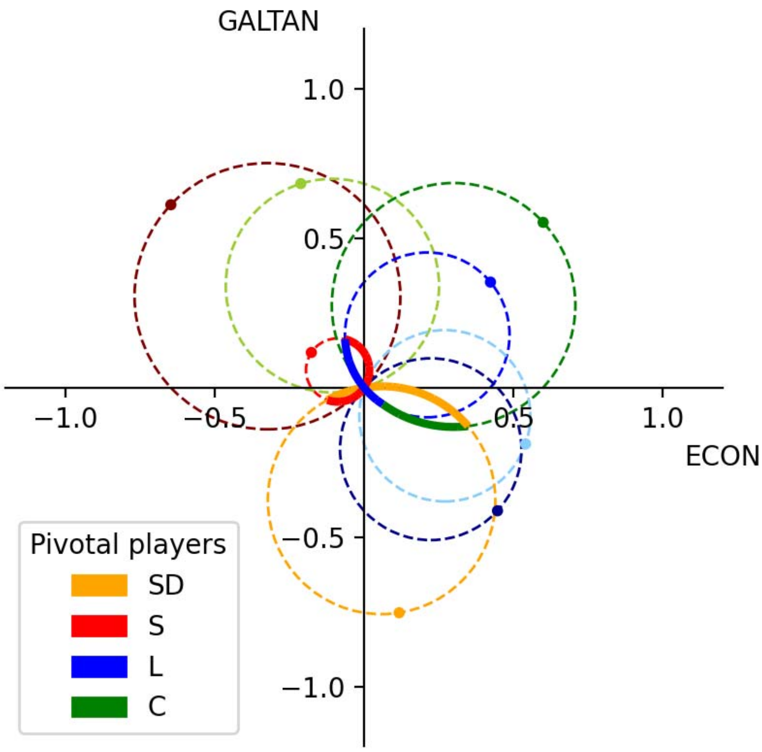

Figure 16: Policy trajectories of individual parties, and the collective policy trajectory, in the Swedish 2022 Riksdag.

[33] The locations of the Swedish parties, following the same order as in Table 1, are approximately $[(-0.18, 0.12), (0.12, -0.75), (0.54, -0.19), (0.60, 0.55), (-0.65, 0.61), (0.45, -0.41), (-0.21, 0.68), (0.42, 0.35)]$.

One sees that S is pivotal in a wide segment of issues, including vertical (cultural value) issues, and not on horizontal (economic) issues, while SD is pivotal on a segment of issues, that includes horizontal issues, but not on vertical issues. Thus one interpretation of this graph is that the other parties, on each side of the economics ideology (left-right) spectrum, will have a hard time pushing their preferred policies on economic issues without the support of SD. Hence, to the extent that they would like to diminish the political power of SD (which they have said), they should settle for centrist economic policies. By contrast, party C is pivotal for a narrow segment of "mixed" issues, around -45^o degrees, issues where right-wing economics attitudes are aligned with authoritarian cultural values. The third largest party in parliament, M, falls in the "shadow" of other parties and is not pivotal on any issue. (See Figure 21 for an explicit representation of the issue segments where different parties are pivotal.)

The next diagram shows once more the collective policy trajectory of the Swedish 2022 Riksdag.

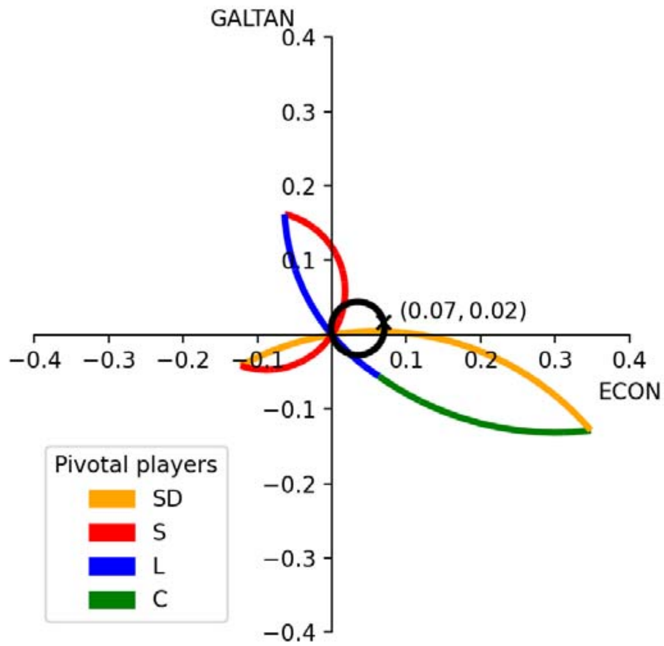

Figure 17: The collective policy trajectory and the projection median of the Swedish parliament. The cross indicates the projection median.

The diagram also shows the parliamentary projection median, indicated by a cross, and the policy trajectory associated with it. If all the eight parties in the parliament, for example in a situation of severe crisis, were to commit to a common platform in ideology space, from which to derive policies for all issues that may arise, and all issues were equally likely, then this would be their projection median, and its policy trajectory would define the policies taken on all issues.

The above diagrams can also be used to calculate the political power of all parties, defined as in Section 5.1, as the probability of being pivotal on a randomly and uniformly drawn issue. Then each party's political power in the parliament is proportional to the angle of its issue segments where it is pivotal, see the diagram below.

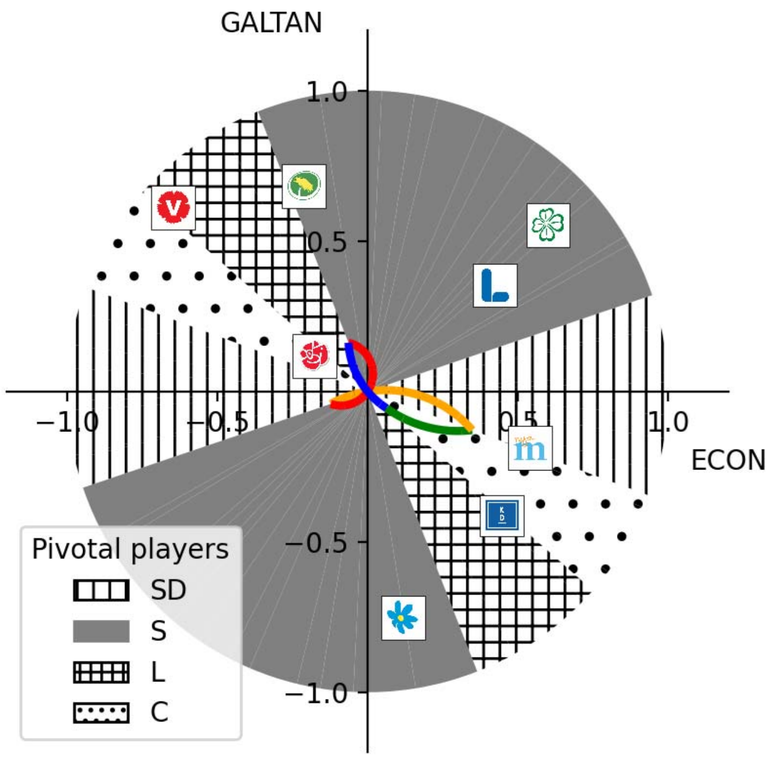

Figure 18: The political power distribution in the Swedish Riksdag.

The resulting numbers are shown in column 1 of the table below. Column 2 shows the spatial Shapley values obtained from another data set (SVT valkompass.svt.se), based on the parties' answers, right before the election, to questions about their positions on 30 different issues. For each issue there were 4 alternative answers (disagree strongly, disagree, agree, agree strongly). We located each party in the corresponding ideology space in $\mathbb{R}^{30}$.[34] Column 3 shows the classical Shapley values (Shapley, 1953).

Table 2: Political power of the parties in the 2022 Swedish Riksdag.

party	power (Chapel Hill)	power (SVT)	Shapley value
S	0.516	0.308	0.381
SD	0.215	0.284	0.238
M	0	0.096	0.238
C	0.118	0.062	0.029
V	0	0.022	0.029
KD	0	0.044	0.029
MP	0	0.023	0.029
L	0.151	0.160	0.029

The numerical results for the three power indices contrast starkly. In terms of the political power measure applied to our 2-dimensional data (shown in Figure 18), parties M, V, KD, and MP have no power at all. The reason is that none of these four parties is pivotal for any issue. In two-dimensional ideology space they, so to speak, stand in the shadow of other parties. By contrast, in terms of the political power measure applied to the 30-dimensional data set, M has approximately one-tenth of the political power in parliament. The other three parties that are powerless in two dimensions have parliamentary power of 2-6% in the thirty-dimensional data set. It is noticeable that L, despite its small number of seats in parliament (16 out of 349), has significant power according to both power calculations, 16%, more than the much larger party M (with 68 seats). In terms of the classical Shapley value, only three parties have significant power in parliament: S, SD and M.

The current government (as of November 2022) consists of 3 formal government members (with ministry portfolios), M, KD and L, and one informal member (with representation in the ministries), SD. If one views these four parties as a coalition, then it has a (weak) majority in parliament and thus political power 1, and the

[34] By treating each issue as a basis vector in $\mathbb{R}^{30}$, we neglected the possibility of "overlap" in line with the famous red-bus-blue-bus paradox in random utility theory.

empowerment of its member parties, in terms of the power measure applied to the 30-dimensional data set, is approximately 0.71.[35] Each member party's political power thus increased by about 71% when forming this majority coalition, as compared with its original political power in parliament. This empowerment is significant, and can be compared with the empowerment of the members of any alternative government coalitions. Had, for example, the other four parties, S+C+V+MP, formed a minority government, assuming that these four parties commit themselves to the ideology location given by their coalition's power center, as in the ex-ante coalition formation approach then that coalition would have obtained power measure 0.440 as applied to the 30-dimensional data set, with an empowerment of only 0.058. We recall that the empowerment index measures the relative change of power that the members of the group obtain as compared to the case where no coalition is formed. In a sense it captures the intensity of the impetus the political agents have to form a given coalition, while it does not capture the incentives for coalition members to break up the coalition and form another coalition, or for non-coalition members to act in their own interest. See Table 3 below for the power and empowerment of eight potential government coalitions that were discussed in the media around the time of the election.

Table 3: Political power and empowerment of eight potential coalitions.

group	power (SVT)	empowerment	seats	main opposition
S+M	1	1.476	175	-
SD+M+KD+L	1	0.711	176	-
S+C+MP+L	0.817	0.478	165	V: 0.075
S+C+L	0.726	0.370	147	SD: 0.073
SD+M+KD	0.491	0.156	160	L: 0.213
S+C+V+MP	0.437	0.051	173	L: 0.291
M+C+KD+MP+L	0.417	0.080	145	S: 0.296
M+C+KD+L	0.403	0.111	127	SD: 0.309

The table shows that among the two majority groups with full power, S+M has the highest empowerment. This means that M could exercise more influence in a coalition with the largest party, as compared to the coalition with the second largest party and two small parties; the current government coalition. Moreover, from the ideology location of party M, the distance to the location of the S+M coalition (according

[35] Applying the *ex-ante* approach to a coalition's joint political platform, one obtains the coalition platform $y = (0.296, -0.332)$ for the present government. (In fact, the *ex ante* approach seems fitting, since the four parties together formulated a joint policy document, the so-called "Tidöavtalet".)

to the ex ante approach) is roughly 0.623, while the distance to SD+M+KD+L is 0.897. Hence, if M is policy motivated and has Euclidean preferences, then S+M is preferable over SD+M+KD+L. We also note that the non-majoritarian coalition with the highest power measure is S+C+MP+L, also has the highest empowerment among those coalitions. According to the table, this is thus a potential government coalition, if the current government would fall and S+M would not form government.

7.1. Voting in the 2022 parliamentary election. We finally apply our framework to analyze the electoral competition between Swedish parties under the 2022 election. For ease of exposition and analysis, we focus on the case of a Gumbel distributed abstention utility and i.i.d. Gumbel distributed valences, where all the Gumbel distributions share the same scale parameter β. This setting was outlined in Section 6.2. The diagram below again shows the platforms of all eight political parties.

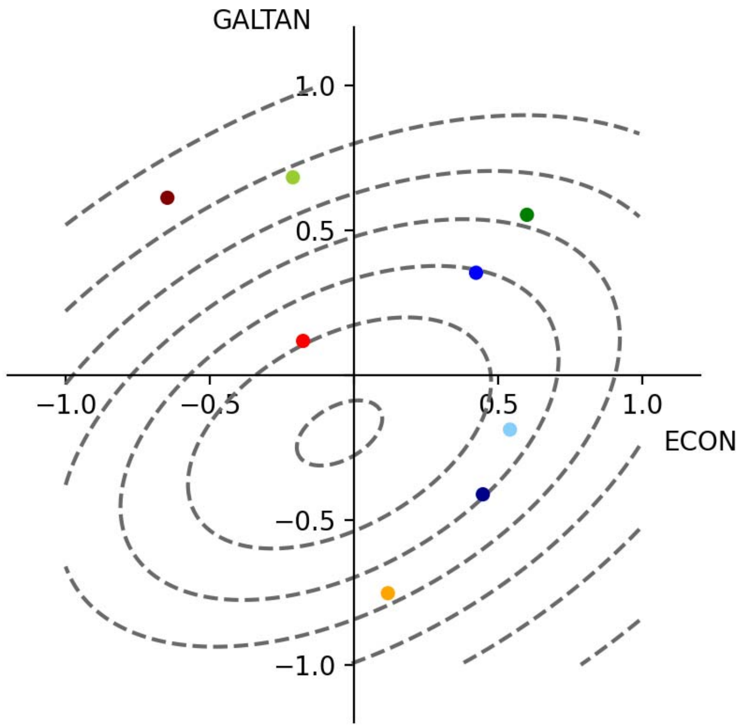

Figure 19: The eight parties in the Swedish Riksdag in a GAL-TAN diagram, and the contour map of the estimated distribution of the Swedish electorate.

The parties to the left of the vertical axis are, from the upper right and counterclockwise, MP, V, and S, while the parties to the right of the vertical axis are, from top to bottom, C, L, M, KD, and SD. The dashed lines make up a contour map for an estimated voter distribution ϕ, based on the actual voting outcome under the assumption that ϕ is bivariate normal. The parameter values for the voter distribution were estimated simultaneously with the scale and location parameters for the Gumbel distributions. The estimated mean value and covariance matrix for the voter distribution are approximately

$$x^o = (-0.05, -0.2) \text{ and } \Lambda = \begin{bmatrix} 0.825 & 0.278 \\ 0.278 & 0.475 \end{bmatrix} \tag{53}$$

The voter distribution being centrally symmetric (by assumption), x^o is also its projection median. The estimated common scale parameter β for the Gumbel distributions was approximately 0.005. This gives a very steep increase in the abstention rate as a function of the distance to the parties' platform profile. As seen in Figure 22, these distances are of the order of magnitude of 1. The estimated difference between the location parameter ν_1 of the common valence distribution for all eight parties and the location parameter ν_0 of the abstention utility distribution was approximately 1.095.

The purpose of this numerical exercise is merely to illustrate how the framework can be used, without claiming that the reported numerical results have high empirical validity. For this, more data and more statistical analysis are required. The actual and estimated votes $v_i(\vec{y})$, along with the turnout $T(\vec{y})$, are given in the Table 4 below.[36]

Figure 20, below, shows a contour map of the abstention rate, $q_0(\vec{y}, x)$, at all voter locations $x \in X$, based on the estimated parameters. The small circular dots are the locations of the parties, and the abstention rate increases very sharply from virtually zero (approximately e^{-20}) at the second largest contour level to about 99% at the outer-most isoquant.

[36] The actual turnout in the 2022 Swedish general election is actually 84.2%. However, roughly 1.05% of the votes cast were invalid and 0.35% of them were cast for parties that didn't reach the 4% threshold for representation in the parliament.

Table 4: Actual and estimated vote outcomes in the 2022 Riksdag election.

	actual votes	**estimated votes**
V	0.056	0.067
S	0.255	0.225
MP	0.043	0.022
C	0.056	0.067
L	0.039	0.056
M	0.161	0.128
KD	0.045	0.083
SD	0.173	0.163
Turnout	0.828	0.809

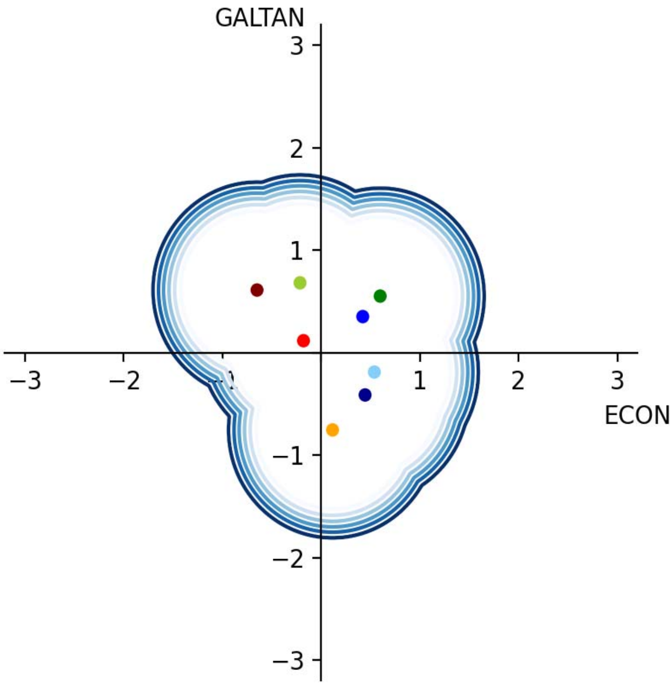

Figure 20: Contour map for the abstention rate in the Swedish 2018 Riksdag election.

We conclude by running a thought experiment and ask, if SD, the party that entered the parliament most recently (in 2005), were free to choose its platform, and would do so in order to maximize its vote share (hence mandates), where would it locate itself? In order to answer that question within the present framework, and with the limited data we have, we suppose that the platforms of all other parties in the GAL-TAN diagram were as reported in 2019 (see the above diagram), and that the voter distribution ϕ was the same as estimated for 2022.

Below is the contour map of vote share of the new entrant.

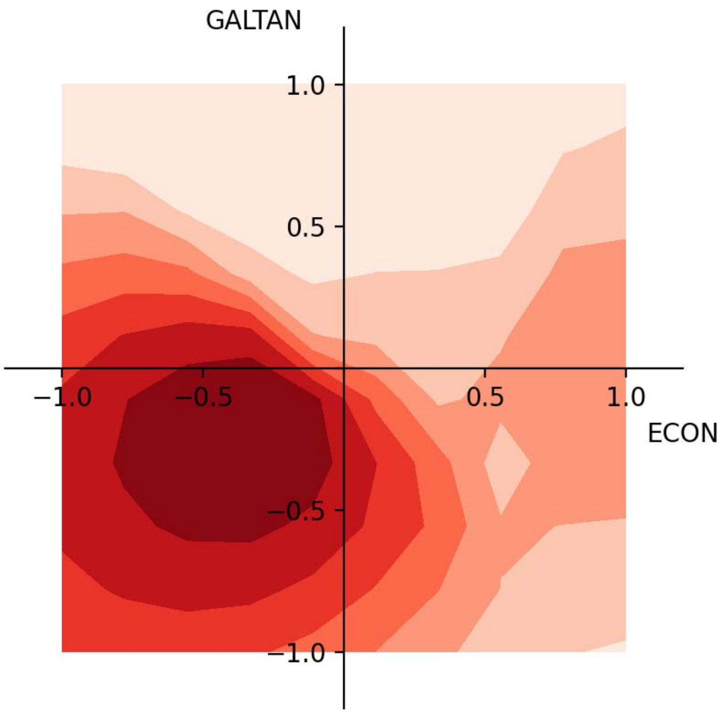

Figure 21: Contour map for the vote share to an entrant party, given the platforms of the seven traditional parties, under i.i.d. Gumbel distributed valences.

The darker-colored regions correspond to higher vote shares. One sees that the entrant would maximize its relative vote share if it would locate roughly at $\hat{y} = (-0.45, -0.4)$. The current location of SD is $y = (0.12, -0.75)$. It is noteworthy that the current platform of SD is not far from the optimal platform for a purely vote-share maximizing new entrant party. The incentive for initiating a new party is, arguably, the strongest in those parts of the electorate where a new party platform would attract the highest vote share. An important driving force is then abstention due to alienation. We also note that in the two-year period 2017-19, the SD party moved from $y' = (0.19, -0.79)$ to its current location, that is, in the direction towards $\hat{y}$. The platform of the biggest party among the seven "old" parties, S, is the closest among these to the optimal location for a vote-share maximizing entrant party. Hence, such an entrant would presumably have fought with S over "swing" voters, those located roughly halfway between S and $\hat{y}$. The current location of SD is more to the right in the GAL-TAN diagram, closer to KD and M than to S.

8. Discussion

The framework presented here enables unified modeling and analysis of political phenomena within a class of multidimensional public choice settings that includes decision-making in parliaments and committees, and that also includes electoral competition. We here modeled four such settings, or forms of government, as we call them. Evidently, an interested modeler can specify other social-choice procedures and analyze them within the same framework.

Much of the focus here is on how to model representative democracy. Arguably, voter abstention is an important factor in many real-life representative democracies. When allowing for abstention, the objectives of political candidates and political parties matter. The objective to maximize votes is then distinct from the objective to maximize vote shares, and the latter is arguably more relevant in many settings, including not only elections to parliaments with proportional representation, but also one-round presidential elections, or the second round in two-round such elections. While it is true that the winner of the office in question is the candidate with most votes, it may be suboptimal for a candidate A to strive to obtain as many votes as possible against given political platforms of the competitors, simply because A's choice of platform will also influence abstention and the number of votes on the competitors.

To the best of our knowledge, this study is among the first to analyze competition between vote-share maximizing political candidates or political parties when voters may abstain from voting due to alienation. Hinich and Ordeshook (1969), Ledyard (1984), and Llavador (2000) analyzed models with abstention due to alienation, but they did not analyze competition between vote-share maximizing political agents. Hinich and Ordeshook analyzed competition between two plurality-maximizing candidates in multidimensional ideology space, Ledyard studied a finite electorate that is given two fixed alternatives to vote for, as in the committee setting. Llavador, finally, analyzed competition between two candidates in unidimensional ideology space, both for candidates that maximize the probability from winning, and for policy-motivated candidates. Wittman (1983), Calvert (1985), and Alesina (1988) analyzed electoral competition between two policy-motivated candidates in a probabilistic setting. However, none of these studies allowed for voter abstention due to alienation. In Wittman (1983) and Calvert (1985) the uncertainty about the outcome of the election stemmed from incomplete information. Alesina (1988), in a setting with infinitely repeated elections, made certain assumptions about the relationship between probability of winning an election and the candidates' strategies without incorporating explicit microfoundations. Also Duggan and Fey (2005) analyzed the implication of policy-motivated candidates for the existence of equilibria when competition between two political agents is multidimensional. However, the analysis takes place in a deterministic setting with finitely many voters. Krasa and Polborn (2012) considered two vote-share maximizing candidates, but did not allow for abstention due of alienation.

Our framework also permits analysis of coalition formation. Arguably, this is another big topic, of particular relevance for government formation in split parliaments. The framework developed here enables analysis of coalition formation, not only for enhancement of political power but also for more forcefully influencing subsequent policy outcomes.

The suggested framework is based upon many heroic and unrealistic assumptions. However, the assumptions together enable mathematical analysis of a wide range of topics. For each result obtained in the present study, one may ask how much, if at all, the underlying assumptions can be relaxed without affecting the claims, and also how claims will change under weaker or alternative assumptions. We here briefly discuss four strong assumptions that underlie most results.

One strong assumption is that voters and political parties have quadratic (one-dimensional) preferences over policies on each issue. Thus, a downward deviation in policy is deemed equally beneficial or harmful as an upward deviation. A relaxation of this symmetry assumption is feasible but seems analytically challenging.[37] Moreover, in part of the analysis, we assumed that disutilities were given equal weight for all issues. In that special case, all issues are "equally important" for every voter. Under the current assumptions in this framework, if issues are uniformly randomly drawn, then voters' derived preferences over political platforms become Euclidean, while for other issue distributions, voters' derived preferences over political platforms are non-Euclidean.

[37] For an individual issue, this assumption can be relaxed to single-peaked preferences. However, with uncertainty concerning issues, the shape of disutility functions for the one-dimensional policies for each issue does matter.

A second strong assumption is that issues may be regarded as straight lines, and that the policy position prescribed by an ideology on any issue is given by the orthogonal projection of that ideology onto the line that represents that issue. This abstraction, due to Owen (1971) and Shapley (1977), is unusual and may appear far-fetched. However, it is hard to think of a more straightforward way of obtaining basis-free multidimensional representation of political decision-making in higher dimensions than one. It is also arguably the most natural generalization of the standard framework with finitely many issues, where each issue is regarded as an axis of a Euclidean space.

A third strong assumption is that the probability distribution of issues is exogenous. In practice, part of political life, also inside parliaments, is to act as an agenda setter, to influence what issues will be on the table. Usually, this is easier for those political parties in parliament who have formed the government. This would have the effect of increasing the power of a party coalition that forms government beyond the power calculated within the present framework. One potential way to endogenize the probability distributions of issues is to let it depend positively on the distribution of mandates in parliament, and on what issues the parties and coalitions in parliament find important. Data on the importance that political candidates or parties place on different issues is sometimes available, such as in political "compasses" where not only positions on political issues are asked for but also the importance attached to them.

A fourth strong assumption in the present analysis is that individual members in parliament always vote according to the "party line", the ideology location of the party that they represent. Evidently this is not always the case in real parliaments. A richer model of voting in parliaments is therefore called for. Potentially the present framework allows for such generalization, by way of applying its approach to coalition formation among political parties in a parliament to coalition formation among individual members of parliament. An avenue for future work. In the same vein, one might model the formation of political parties, in the first place, as *ex-ante* coalitions formed between citizens before an election. However, while a political party may have political power also without joining a coalition in parliament, this is not the case for individual citizens in the present framework, where the electorate is represented as a continuum. A more appropriate representation would then be to treat the electorate as finite and analyze the formation of political parties along the same lines as coalition formation between political parties, where now each citizen has an ideology location. Another avenue for future work.

An important topic that has not been addressed here is non-proportional representation in legislative bodies. In many countries, including the UK and the USA, a particular seat in the legislative body in question may be reserved for a representative for a particular geographical district, and the representative for that district may be elected in the district according to majority rule. Thus, a political minority of the total electorate (say, democrats or republicans in the US) may obtain a majority in the legislative body (say, the house of representatives). Future work in such a direction would be desirable.

One caveat is in place here concerning applications of the framework to real-life data. Extra care is needed to identify the axes, the base issues, of the ideology space, since the framework is vulnerable to the "blue-bus-red-bus" paradox, known from discrete-choice random utility theory. In the present framework, the paradox is that if two bases are essentially the same for all voters — so to speak represent the same "underlying issue" — and if one assumes that issues are uniformly randomly drawn, then the underlying issue becomes more frequent than other issues in decision making, political power and voting behavior in elections. The present framework is silent on the choice of axes when defining the ideology space. A promising direction seems to be to base the choice of axes on empirical results concerning principal components of real-life political preferences. See e.g. Alós-Ferrer and Granić (2015), where four principal components were identified. Such principal components can be used as a basis for ideology spaces, and then the red-bus-blue-bus paradox is avoided.

We hope that this framework will be helpful for future theoretical and empirical research on the workings of democratic institutions.

9. Appendix

For much of the analysis in the proposed framework, it is convenient to identify pairs of mirror unit vectors $\theta, -\theta \in \Theta$ as one equivalence class, $\theta \sim -\theta$, and without loss of generality fix a unit *hemisphere* that is isomorphic to the quotient space $\Theta/\sim$. To do this, we use spherical coordinates.

In the special case $d = 2$ this corresponds to the usual polar coordinates; any point $x \in X$ is represented as a pair $x = (r, \alpha)$, where $r \in \mathbb{R}_+$ is the *radial coordinate*, the distance from the origin and $\alpha \in [0, 2\pi)$ the *angular coordinate*, the angle with respect to the horizontal axis, such that the Cartesian coordinates can be recovered as follows: $x_1 = r\cos\alpha \quad \text{and} \quad x_2 = r\sin\alpha$. The unit circle can then be written as

$$\Theta = \{(r, \alpha) \in X : r = 1 \text{ and } \alpha \in [0, 2\pi)\} \tag{54}$$

and the *unit semicircle* Θ' as

$$\Theta' = \{(r, \alpha) \in \hat{\Theta} : \alpha \in [0, \pi)\} \tag{55}$$

In higher dimensions ($d > 2$), the unit sphere is defined in spherical coordinates by

$$\Theta = \{(r, \alpha_1, \ldots, \alpha_{d-1}) \in X : r = 1,\ \alpha_j \in [0, \pi) \text{ for } j = 1, \ldots, d-2, \text{ and } \alpha_{d-1} \in [0, 2\pi)\} \tag{56}$$

and the *unit hemisphere* then is

$$\Theta' = \{(r, \alpha_1, ..., \alpha_{d-1}) \in \Theta : \ \alpha_j \in [0, \pi) \text{ for } j = 1, \ldots, d-1\} \tag{57}$$

9.1. Proof of Lemma 2. Let $x, y \in X$. We first note that any two points lie on a line, and since all concepts and the analysis is invariant with respect to translation and change of basis, we can, without loss of generality, fix a basis so that x is the origin, and $y = (a, 0, ..., 0)$ for some $a \in \mathbb{R}$. For any issue $\theta \in \Theta$, the ideal policies of these locations are $t_\theta(x) = 0$ and $t_\theta(y) = a\cos\alpha_1$, where $\alpha_i \in [0, 2\pi)$ is the angle between the unit vector and ith axis. Then:

$$\int_{\theta\in\Theta} \bar{\lambda} \left| t_\theta(x) - t_\theta(x) \right|^2 d\kappa(\theta) = \bar{\lambda} a^2 \frac{\int_0^{2\pi} \ldots \int_0^{2\pi} \cos^2 \alpha_1 d\alpha_1 \ldots d\alpha_{d-1}}{\int_0^{2\pi} \ldots \int_0^{2\pi} 1\alpha_1 d\alpha_1 \ldots d\alpha_{d-1}} = \bar{\lambda} a^2 \frac{\pi}{2\pi} = \frac{\bar{\lambda}}{2} \left\| x - y \right\|^2 \tag{58}$$

9.2. Proof of Lemma 3. Let $G : \Theta \times \mathbb{R} \to [0, 1]$ be defined by $G(\theta, t) = \Phi_\theta(t) = \int_{\theta \cdot x \le t} \phi(x)\, dx$. Since ϕ is continuous, so is G. By definition, $G(\theta, t) = \frac{1}{2}$ at $t = \tilde{t}$. Moreover, $\lim_{t\to-\infty} G(\theta, t) = 0$ and $\lim_{t\to+\infty} G(\theta, t) = 1$. For any $\varepsilon \in \left(0, \frac{1}{2}\right)$, let $t_\varepsilon > \tilde{t}_\theta$ be such that $G(\theta, t_\varepsilon) = \frac{1}{2} + \varepsilon$, and let $t_{-\varepsilon} < \tilde{t}_\theta$ be such that $G(\theta, t_{-\varepsilon}) = \frac{1}{2} - \varepsilon$. By continuity of $G(\theta, t)$, there exists a $\delta > 0$ such that for all $\theta' \in \Theta$ with $\|\theta' - \theta\| < \delta$, $G(\theta', t_\varepsilon) > \frac{1}{2}$ and $G(\theta', t_{-\varepsilon}) < \frac{1}{2}$. Again by continuity, there exists a $t' \in (t_{-\varepsilon}, t_\varepsilon)$ such that $G(\theta', t') = \frac{1}{2}$. By definition, $\tilde{t}_{\theta'} = t'$, which establishes that $\tilde{t}_\theta$ is a continuous function of θ. Hence also $\tilde{x}_\theta = \tilde{t}_\theta \theta$ is a continuous function of $\theta \in \Theta$. This establishes the claimed continuity of $\theta \mapsto \tilde{x}_\theta$. Second, since this function is continuous, and Θ is a compact and connected set, so is its direct image $\tilde{X}$.

9.3. Proof of Proposition 2. Without loss of generality (since all concepts are translation invariant), let x^o be the origin in $X = \mathbb{R}^d$. From Proposition 2 we know that if there would be no abstention, the ideal policy $t_\theta(x^o) = 0$ from the vantage of x^o is the median-voter policy $\tilde{t}_\theta$ and hence the Condorcet winner for every issue $\theta \in \Theta$. This is also true under (endogenous) abstention.

To see this, let $\theta \in \Theta$ be any issue, let ϕ_θ be the probability density induced by ϕ on the straight line L_θ through θ,

$$\phi_\theta(t) = \frac{\partial \Phi_\theta(t)}{\partial t} \quad \forall t \in \mathbb{R}$$

and consider a voting contest on this issue between policy $t_\theta(x^o) = 0$ and a policy $t > 0$. For every $z > 0$, the induced density at $t/2 - z$ exceeds that at $t/2 + z$, while the abstention rate is the same at all locations $x \in X$ with ideal policy $t/2 - z$ (on issue θ) as at all locations $x \in X$ with ideal policy $t/2+z$ (on issue θ). This implies that, among those who vote, more than half the voters prefer $t_\theta(x^o)=0$ over t:

$$\int_0^{+\infty} \phi_\theta\left(\frac{t}{2}-z\right)\left[1-a\left(\frac{t}{2}-z\right)\right]dz > \int_0^{+\infty} \phi_\theta\left(\frac{t}{2}+z\right)\left[1-a\left(\frac{t}{2}+z\right)\right]dz \tag{59}$$

where $a\,(t/2-z) = a\,(t/2+z)$ are the mentioned abstention rates. See diagram below, where $d=2$, $\theta=(2,1)/\sqrt{5}$ and $t=\sqrt{5}$.Thus the median policy $t_\theta(x^o)=0$ is a Condorcet winner on any issue $\theta \in \Theta$ against any policy $t>0$. By symmetry, it holds for all contestant policies $t<0$. Moreover, by the same argument, no other policy is a Condorcet winner: for any candidate policy $t \neq t_\theta(x^o)$, let $t_\theta(x^o)=0$ be a challenger.

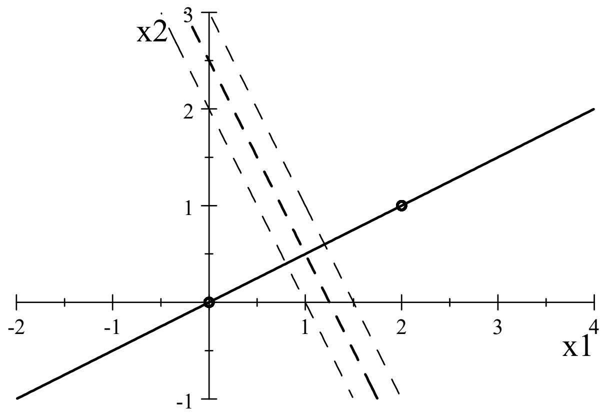

Figure A1: The geometry in the proof of Proposition 2.

9.4. Proof of Proposition 4. Let $x^o \in X$ be the mode of ϕ, let $y \neq x^o$, let $\vec{y} = (x^o, y)$, and let $X_1(\vec{y})$, $X_2(\vec{y})$, H, and $g: X \to X$ be as in the proof of Proposition 3. For two political agents with platform profile $\vec{y}$ and with the same valence distribution, (42) implies that the abstention rates at mirror points are equal; $q_0(\vec{y},x) = q_0(\vec{y},g(x))$. Moreover, conditional on voting (that is, not abstaining), the probability that a voter at a location $x \in X$ votes for an agent is equal to the probability that a voter at $x' = g(x)$ votes for the competitor. Together, these observations imply, in force of (43), that $q_1(\vec{y},x) = q_2(\vec{y},g(x))$. Again voters with locations in H have Lebesgue measure zero, so they do not affect the vote shares. Rewriting (44) as

$$v_i(\vec{y}) = \int_{X_1(\vec{y})} q_i(\vec{y},x)\phi(x)dx + \int_{X_1(\vec{y})} q_i(\vec{y},g(x))\phi(g(x))dx \tag{60}$$

one obtains that the agent at x^o receives more votes than the agent at y:

$$v_1(\vec{y}) - v_2(\vec{y}) = \int_{X_1(\vec{y})} [q_1(\vec{y}, x) - q_2(\vec{y}, x)] (\phi(x) - \phi(g(x))dx > 0 \tag{61}$$

where the first factor in the integrand is positive since $\|x - x^o\| < \|x - y\|$ for all $x \in X_1$, and the second factor is positive since ϕ is unimodal and centrally symmetric around x^o. So $y \neq x^o$ is not a best reply to x^o in terms of vote shares. The unique best reply to x^o is x^o itself. At the alternative platform profile, $\vec{y} = (x^o, x^o)$, both agents obtain their value of this constant-sum game, 1/2. By the same argument, no other policy platform profile than $\vec{y} = (x^o, x^o)$ is an equilibrium.

9.5. Proof of Proposition 3. Let $x^o \in X$ be the mode of ϕ. Suppose that political agent 1 has taken platform x^o, and consider any platform $y \neq x^o$ for political agent 2. Since the agents have the same (deterministic) valence, the platform profile $\vec{y} = (x^o, y)$ divides the electorate in two parts, separated by the hyperplane H (of indifferent voters) through the midpoint $\frac{1}{2}(y + x^o)$ and orthogonal to the vector $y - x^o$; $H = \{x \in X : \|x - y\| = \|x - x^o\|\}$. Then $X_1(\vec{y})$ is the (closed) half-space that contains x^o and $X_2(\vec{y})$ the (closed) half-space that contains y. Voters in $H = X_1(\vec{y}) \cap X_2(\vec{y})$ have Lebesgue measure zero, so they do not affect the outcome of the election. Since ϕ is symmetric and unimodal around x^o, there are more voters in $X_1(\vec{y})$ than in $X_2(\vec{y})$.

Let $g : X \to X$ be the bijection that maps each point $x \in X$ to its "mirror" point, or "reflection", with respect to the affine hyperplane H: $g(x) = x + \delta(y - x^o)$ for $\delta \in \mathbb{R}$ such that $x + \frac{\delta}{2}(y - x^o) \in H$. Clearly reflection is a symmetric relation; if $x' = g(x)$ then $x = g(x')$, for all $x \in X$. Moreover, $\|x - x^o\| = \|g(x) - y\|$ for any $x \in X$. Since the two political agents have the same valence, (36) implies that the utility from voting is equal for voters at locations x and $x' = g(x)$; $\hat{U}(\vec{y}, x) = \hat{U}(\vec{y}, x')$, and thus $F_0[\hat{U}(\vec{y}, x)] = F_0[\hat{U}(\vec{y}, x')]$. Moreover, F_0 has full support and ϕ is unimodal and centrally symmetric around x^o, so agent 1 obtains more votes than agent 2:

$$v_1(\vec{y}) - v_2(\vec{y}) = \int_{X_1(\vec{y})} F_0[\hat{U}(\vec{y}, x)](\phi(x) - \phi([g(x)])dx > 0 \tag{62}$$

Thus $s_2(\vec{y}) < 1/2$, and therefore $y \neq x^o$ is not a best reply to x^o in terms of vote shares; the unique best reply to x^o is x^o itself. At platform profile $\vec{y} = (x^o, x^o)$ both agents obtain their value of this constant-sum game, 1/2. By the same argument, no other policy platform profile than $\vec{y} = (x^o, x^o)$ is an equilibrium.

9.6. Proof of Proposition 5. Without loss of generality, let $\bar{\lambda} = 2$. Each of the random variables $U_i(\vec{y}, x)$, defined in (35), is $Gumbel\left(\nu_i - \|x - y^i\|^2, \beta\right)$ distributed.

In line with (36), let $\hat{U}(\vec{y}, x)$ be the random utility that a voter at location x achieves from actively voting (that is, when abstention is not an option). As is well-known in the random-utility literature, also this random variable is Gumbel distributed, with CDF $\hat{F}$ defined for all utility levels $s \in \mathbb{R}$ by

$$\begin{aligned} \hat{F}(s) &= \prod_{i \in I} \exp\left[-e^{-\left(s-\nu_i+\left\|x-y^i\right\|^2\right)/\beta}\right] = \exp\left[-\sum_{i \in I} e^{-\left(s-\nu_i+\left\|x-y^i\right\|^2\right)/\beta}\right] \quad (63) \\ &= \exp\left[-e^{-s/\beta} \sum_{i \in I} e^{\left(\nu_i-\left\|x-y^i\right\|^2\right)/\beta}\right] = \exp\left[-e^{-(s-\hat{\nu}(\vec{y},x))/\beta}\right] \end{aligned}$$

where

$$\hat{\nu}(\vec{y}, x) = \beta \ln\left(\sum_{i \in I} e^{\left(\nu_i-\left\|x-y^i\right\|^2\right)/\beta}\right) \quad (64)$$

The probability that a voter located at $x \in X$ will abstain from voting is thus

$$\begin{aligned} \Pr\left[\hat{U}(\vec{y}, x) \leq \varepsilon\right] &= \int_{-\infty}^{+\infty} \exp\left[-e^{-(s-\hat{\nu}(\vec{y},x))/\beta}\right] dF_0(s) \quad (65) \\ &= \int_{-\infty}^{+\infty} \exp\left(-e^{-s/\beta} \sum_{i \in I} e^{\left(\nu_i-\left\|x-y^i\right\|^2\right)/\beta}\right) dF_0(s) \end{aligned}$$

The probability that the voter will vote for candidate i is

$$\begin{aligned} q_i(\vec{y}, x) &= \Pr\left[U_i(\vec{y}, x) = \hat{U}(\vec{y}, x) \ \wedge \ \hat{U}(\vec{y}, x) > \varepsilon\right] \quad (66) \\ &= \Pr\left[U_i(\vec{y}, x) = \hat{U}(\vec{y}, x) \mid \hat{U}(\vec{y}, x) > \varepsilon\right] \cdot \Pr\left[\hat{U}(\vec{y}, x) > \varepsilon\right] \end{aligned}$$

By the "invariance of the maximum" property, Fosgerau et al. (2018), which holds for a class of probability distributions that includes the Gumbel distribution,

$$\Pr\left[U_i(\vec{y}, x) = \hat{U}(\vec{y}, x) \mid \hat{U}(\vec{y}, x) > \varepsilon\right] = \Pr\left[U_i(\vec{y}, x) = \hat{U}(\vec{y}, x)\right] \quad (67)$$

Finally, is well-known from random-utility theory that

$$\Pr\left[U_i(\vec{y}, x) = \hat{U}(\vec{y}, x)\right] = \frac{e^{\left(\nu_i-\left\|x-y^i\right\|^2\right)/\beta}}{\sum_{j \in I} e^{\left(\nu_j-\|x-y^j\|^2\right)/\beta}} \quad (68)$$

This establishes (43), and the other claims follow.

9.7. Proof of Proposition 6. By Lemma 2, the assumed uniformity of the issue distribution implies that agent i's goal to minimize its loss function (50) is equivalent with minimization of the distance $\|y^* - x^i\|$ between the winning political platform $y^* \in X$ and the agent's ideology location $x^i \in X$. Moreover, as shown in the proof of 4, the symmetry and unimodality of the voter distribution ϕ implies that if the agents' political platforms are at different distances from the median voter's location x^o, then the agent with political platform closest to the median voter obtains a strict majority of the votes given. If their platforms are at equal distance from x^o, there will be a tie. Without loss of generality, set $x^o = \mathbf{0}$.

First, suppose that $y^2 = \mathbf{0}$. As shown in the proof of Proposition 4, agent 1 will then obtain less than half the votes given, for all platforms $y^1 \neq \mathbf{0}$. Thus, any strategy profile $\vec{y}$ where $y^i = \mathbf{0}$ for $i = 1$ or $i = 2$, or both, is a Nash equilibrium. The policy outcome of all these Nash equilibria is the median voter's ideal policy, for all issues: $y^* = x^o$.

Second, suppose that $x^1 = \delta x^2$ for some $\delta > 1$, and let $y^2 = x^2$. Then no platform y^1 with $\|y^1\| < \|y^2\|$ is a best reply for agent 1, since although 1 would obtain more than half the votes given, the ensuing policy decisions in parliament would on average (under the uniform issue distribution) deviate more from 1's ideal policy than had instead the platform of agent 2 been implemented. However, the platform $y^1 = y^2$ is a best reply for agent 1, and so is every platform y^1 with $\|y^1\| > \|y^2\|$, since in all these latter cases agent 2 will win a majority and hence $y^* = y^2$ will be implemented. The same conclusion holds, with reversed roles for the agents, if $x^2 = \delta x^1$ for some $\delta > 1$. If $\delta = 1$, then $y^* = y^1 = y^2 = x^1 = x^2$. In sum, if $x^1 = \delta x^2$ for some $\delta > 0$, then $y^* = \arg\min_{x \in \{x^1, x^2\}} \|x - x^o\|$.

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